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With 150 Figures

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Preface

General

Oscillations are extremely important in all areas of human activities, for all sciences, technologies and industrial applications. Any development, any change can be interpreted as motion. Any motion is deeply connected with one of the most fundamental properties of nature – its ability to react with oscillations at any internal change or external influence.

Sometimes these oscillations are harmless, often they can be noticed as noise or cause wear. Vibrations, if they are not desired, can be dangerous. But sensibly organized and controlled vibrations may be pleasant (think of all kinds of music) or vitally important (heartbeat). If the oscillations are sufficiently small and the considered dynamical system is smooth, it can be very helpful to linearize it in the vicinity of some static or dynamic solution and use the corresponding powerful analytical methods. But in many practical cases the oscillations are either not small or the system is not smooth. In these cases a non-linear analysis becomes essential to understand physics of the system or of the process.

"To understand"- is the key word here. The main objective of any analysis is comprehension. The only systematic way to solve any problem or to improve any system is to understand its nature and foresee its behavior. There are different ways to obtain the knowledge of the object one is working with. The first and still the most important way is the experiment. It is the only way to get any objective information about nature. But this way is not a simple one. The problem is not only that it is often very difficult and expensive to perform real experiments. Even more important is the fact, that the measurement itself never helps a scientist (or an engineer) to understand anything, before he tries to interpret it. And the only way to interpret anything is modeling. However the experiment always remains the last proof of each theory or model. So like two eyes are necessary for spatial vision, both modeling and experiment are inevitable for physical understanding.

Different types of modeling are often used in modern fundamental and applied sciences. The main rule says: Models must be as simple as possible and as accurate as necessary. In practice it means, a simple analytic model is often better than complex numeric one, if it is sufficiently accurate. On the other hand, if simple models are not able to describe significant properties of a natural object, the numeric or even the natural modeling (for example in hydro- and aerodynamics) are the only efficient way to achieve comprehension.

The same statement can be applied not only to natural sciences, but even to mathematics. However, the experiments here are mostly numeric. The greatest

mathematician of the late 19th and early 20th century, Henri Poincaré, wrote: "Usually an equation was considered to be solved if the solution had been expressed in a finite number of known functions. But this is only possible in one case out of hundred. What we always could do, or better should do, is to solve the problem qualitatively. This means trying to find the general form of the curve which tracks the unknown function." (Henri Poincaré, Science et Méthode).

This idea of Poincaré gave the main direction to the development of applied mathematics and mechanics in the 20th century, and it did not loose its actuality even now. His methods to find periodic solutions of the perturbed ordinary differential equations and to analyze their stability [98], caused a mental revolution both in the theory of differential equations and in the nonlinear mechanics. It was the starting shot for the development of the perturbation methods, being now one of the most powerful groups of analytic methods in the theory of nonlinear oscillations.

It is interesting that the Poincaré's statement is also valid for applications. Engineers are seldom interested in particular solutions. Usually the parameters of the system are known with some grade of uncertainty and the initial conditions are very difficult to control. Hence an engineer is interested first of all in general tendencies in the behavior and evolutions of a system, he is working on. To explain the main qualitative phenomena and to predict the qualitative influence of specific parameters is the task of an analyst and the base for design. Thus approximate analytic methods are both necessary and useful.

Nonlinear Oscillations

Dozens of brilliant books were written on different aspects of vibrations both from theoretical and practical points of view. At least the classical works by Lord Rayleigh [108], Poincaré [98] and Timoshenko et al. [126] should be mentioned here. Even these three titles demonstrate the main difficulty for students and practical engineers to find an appropriate balance between sophisticated mathematical methods and problems, which are relevant for applications.

There are excellent textbooks and monographs devoted to different mathematical aspects of nonlinear oscillations. Bogoliubov and Mitropolskii [23], Bolotin [25], Nayfeh [79], Arnold [8], Sanders and Verhulst [114], Guckenheimer and Holmes [44], Mitropolskii and Nguyen [75], Nayfeh and Mook [80], Volosov and Morgunov [134] and Verhulst [131] can be mentioned here. Being significant contributions to the development of mechanics and approximate methods in the theory of ordinary differential equations and dynamical systems, these books remain almost unavailable for the majority of mechanical engineers, first of all, because of their mathematical language.

Books intended for applied scientists and practical engineers are very seldom. Most of them are concentrated on qualitative discussion of practically important particular effects without addressing generality and mathematical rigor of the used methods. The most versatile book of this type is no doubt Panovko and Gubanova [85]. But it is also necessary to point to excellent specialized monographs on oscillations in systems with collisions [10, 15, 58, 62, 77], systems with friction [49,

104, 130], machines with high frequency vibrational excitation [15, 17, 20, 78], autoparametric resonance [129] and chaotic vibrations [76].

"Nonlinear Oscillations in Mechanical Engineering"

The recent book is concentrated on the effects connected with the nonlinearities usual in mechanical engineering. These nonlinearities are caused, first of all, by contacts between different mechanical parts. So the main part of this book is devoted to oscillations in mechanical systems with discontinuities caused by dry friction and collisions. Another important source of nonlinearity is caused by rotating unbalanced parts usual in various machines and variable inertias occurring in all kinds of crank mechanisms, for example in combustion engines.

This book is devoted to nonlinear oscillations and is written for advanced undergraduate and postgraduate students, but it may be also helpful and interesting for both theoreticians and practitioners working in the area of mechanical engineering at universities, in research labs or institutes and first of all in the development departments of industrial corporations.

Mathematical Methods in This Book

Perturbation methods, especially averaging and multiple scales are the basic approach used in this book. The objective here is to adopt these methods to the described class of problems and make their use convenient for mechanical engineers without loosing mathematical correctness.

It was also Poincaré who justified the use of divergent series and introduced the concept of asymptotic analysis. The next important step was done in the middle of the 20th century in Russia. Krylov, Bogoliubov, Mitropolskii and their colleagues [23, 24, 66, 74, 75, 111, 134] suggested and justified the averaging method. This method allows the asymptotic analysis not only of stationary or periodic solutions, but it is also suitable for transient processes. However, the first successful attempts to link perturbation methods with classical variation of parameters were done by van der Pol [99] and corresponding ideas can be found even earlier in the works of Lagrange [68]. The book by Bogoliubov and Mitropolskii [23] gave a strong impulse for the further development of asymptotic methods both due to clear mathematical proof of the approach and due to the great variety of considered applications. It is still an interesting and inspiring source for scientists.

Fast and Slow Motions

Averaging method clearly displays the main feature of different phenomena and processes of fundamental and applied interest taking place in mechanical systems, which allows developing and using asymptotic methods in nonlinear oscillations. Very often motions of these systems can be split into some slow evolution and overlaying "fast" vibrations of a high frequency at a small amplitude. These slow motions describing the evolution of the system are, as a rule, of the main interest for the researcher.

Several scientists have used this property as the fundamental background for the development of further methods and approaches. Two of them should be mentioned here. The first one is the method of multiple scales. It was developed by Nayfeh and his colleagues [79, 80]. This method is substantially very close to the averaging method, but due to its straight forward formulation is very popular in physical applications. First of all, the use of multiple scales is very simple for differential equations with partial derivatives, even if its accuracy is not always proved.

The same can be said about the "method of the direct separation of motions". It was originated in the works of Kapitsa [55, 56]. This method was most generally formulated by Blekhman [16 – 21], who also gave numerous examples of its use for different problems in mechanics and physics. The method of the direct separation of motions is even easier to use than the multiple scales. Its accuracy is proved for the most typical cases and it was successfully used for systems described by both ordinary and partial differential equations [19, 20, 53,119 – 125].

The main uncertainty connected with both multiple scales and direct separation of motions is that if the considered problem is even a little bit aside the typical applications the user never knows if the results are correct or not and the method itself does not give any instrument to control its accuracy. Averaging to the contrary contains a clear way for its mathematical validation including sensible accuracy control. If the requirements of the corresponding theorems are not fulfilled in a practical problem, there is always a chance to expand the method's applicability by formulating and proving new theorems. This area was and remains the mathematicians' domain. The author would be very pleased if he could attract their attention to numerous and diversified engineering problems, even though only an infinitesimally small subset of this inexhaustible field is touched in the present book.

Structure

The book is structured as follows. A short **introduction** to the problems under consideration is given in the first chapter. It includes the standard perturbation techniques alongside the usual simplest descriptions of dry friction and collisions between rigid bodies.

Chapter 2 is devoted to vibrations in systems with dry friction. The discussion encloses both the self excited oscillations due to the negative friction gradient and the friction caused flutter alongside the vibration caused transportation.

Oscillations in systems with almost elastic collisions are discussed in **Chapter 3**. The analysis is based on the ideas of the unfolding transformations, which give a clear and transparent framework for analysis of one dimensional systems restricted from one side (for example an oscillator near a wall) and from both sides (for example the same oscillator in a clearance).

Systems with strong energy dissipation are discussed in **Chapter 4**. The main idea here is to separate the dissipative subsystem, which moves in many important cases as a slave of the almost conservative subsystem (at least in the first order approximation). This approach is consequently applied both to systems with strong linear damping and to systems with inelastic collisions.

Chapter 5 is devoted to the problem of the significantly nonlinear resonance, which occurs always, when the power of an exciter is comparable with the energy demand of the machine. It is actually the case in all real machines otherwise their drive would be too powerful and expensive. So the practical importance of the nonlinear resonance can be hardly overestimated.

The basic ideas of analysis and elementary effects in systems subjected to strong high frequency excitation are discussed in **Chapter 6**. Stiffening, softening, biasing alongside smoothing of dry friction are the main effects illustrated by simple examples. Misbehavior of the "optimally" controlled pendulum under the influence of the HF excitation is the advanced example combining several approaches introduced in the previous chapters.

Further development and general analysis of systems subjected to high frequency excitation is given in **Chapter 7**. Especially results concerning systems excited due to oscillating inertial coefficients are relevant for applications.

All the analysis in the present book is based on the appropriately modified averaging. The relevant theorems are formulated and explained qualitatively. Mathematical proofs are given in the **Appendixes**, which can be omitted by readers interested in applications, but are strongly recommended for those interested in the development of theoretical approaches.

Acknowledgements

The book summarizes and supplements the investigations of the author, published in different scientific journals in different languages – firstly in Russian, then in English.

The author is very thankful to his friend and teacher, Professor I.I. Blekhman, who convinced him to write this book. He is also thankful to his former colleagues at the Fundamental Research Department of the "Mekhanobr"-Institute Professor E.B. Kremer, O.Z. Malakhova, Professor R.F. Nagaev and A.V. Pechenev for inspiring and fruitful discussions. These discussions induced the author to start his activities on the analysis of discontinuous systems and encouraged him to try to modify the averaging method in the appropriate manner.

The author would like to express his gratitude to the colleagues from the Department of Solid Mechanics at the Technical University of Denmark especially to Professor J.J. Thomsen for his support and fruitful discussions. The importance of this collaboration can be hardly overestimated. Sections 2.1, 2.4, 3.1, 3.2 and 6.6 are based on our joint research.

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Karlsruhe, June 2005

Alexander Fidlin

Contents

1. Introduction

1.1. Usual Sources of Nonlinearity in Mechanical Engineering

The world around us and we ourselves are inherently nonlinear. The simplest experiment illustrating this statement is an attempt to bend a wooden beam. As long as the load is small, the deflection of the beam is approximately proportional to the applied force. But at some sufficiently large level the beam will simply break. This strong and definitely irreversible change is an elementary example of nonlinear behavior illustrating an important feature enforcing us to formulate the first statement more precisely. The world is nonlinear, but in many cases, if we consider only small influences and changes, the linear approximation is often sufficient to understand, predict and control its behavior.

Nonlinearities and their consequences in the physical and technical world are highly diversified and the development of the corresponding theoretical framework and mathematical language is still in its infancy. We would like to start with several examples demonstrating the most usual sources of nonlinearity in mechanical engineering.

1.1.1 Geometrical Nonlinearities

The first and the simplest one are geometrical nonlinearities arising form pure kinematics. The first example shows the pendulum (*cf.* Fig. 1.1), whose dynamics is governed by the following equation:

$$\ddot{\varphi} + \frac{g}{l}\sin\varphi = 0 \tag{1.1}$$

For small oscillations around the down pointing equilibrium $\varphi = 0$ this equation can be linearized, but if one is interested in large oscillations or even in the rotational motions of the pendulum its nonlinearity becomes significant.

Another example is based on the crank mechanism usual in all kinds of machines (Fig. 1.2). It consists of a rotating rod, which is attached to a fixed point by a spring with stiffness C and free length x_0. We assume that the spring is linear

(the last statement means that the deflection of this spring is proportional to the applied force, irrespective of its magnitude.) The governing equation for this system has the following form:

$$\ddot{\varphi} = \frac{CL\left(\sqrt{L^2 + l^2 - 2Ll\cos\varphi} - x_0\right)}{ml\sqrt{L^2 + l^2 - 2Ll\cos\varphi}}\sin\varphi \tag{1.2}$$

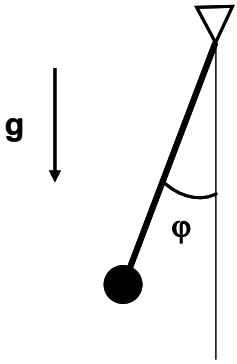

Fig. 1.1. The mathematical pendulum is one of simplest examples of geometrically nonlinear systems

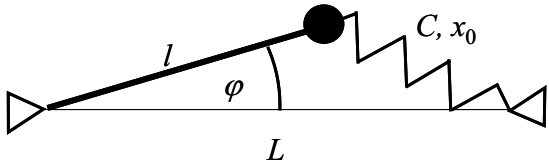

Fig. 1.2. The geometrically nonlinear crank

1.1.2 Physical Nonlinearities

The assumption concerning the linearity of the spring is also correct only for small deflections. Both rubber (Fig. 1.3) and steel (Fig. 1.4) demonstrate nonlinear relationships between stress and strain if the applied load is sufficiently large.

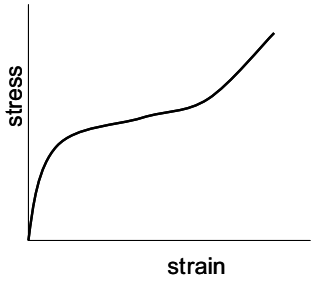

Fig. 1.3. Stress-strain diagram for a rubber like material

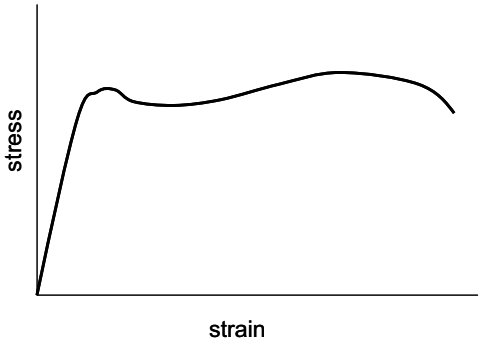

Fig. 1.4. Stress-strain diagram for steel

These two diagrams are the most usual examples of physical or material nonlinearity in mechanical engineering.

1.1.3 Structural or Designed Nonlinearities

In numerous applications the nonlinear characteristic of a spring is desired. Two simple examples of the designed nonlinearities are shown in Fig. 1.5 and 1.6.

Fig. 1.5. Designed hard nonlinearity

The stiffness of the system of springs in the first example increases as the deflection exceeds a certain value x_0, after which the spring on the right hand side of the mass gets active. This behavior is usually called hardening and is sometimes

described as a progressive stiffness characteristic. It is achieved here through the designed clearance between the two springs.

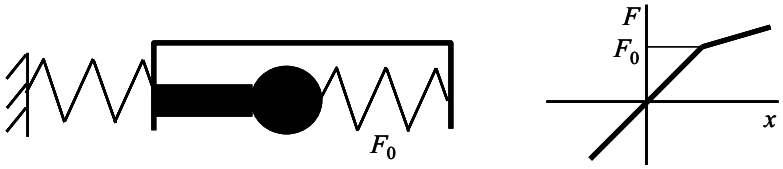

Fig. 1.6. Designed soft nonlinearity

The stiffness of the system of springs in the second example decreases as the load exceeds a certain value F_0 after which the spring on the right hand side of the mass gets active (as long as the external force is smaller that the preload F_0, the spring on the right hand side of the mass presses it against the stop and the whole frame moves as a solid body.) This behavior is usually called softening.

These two examples belong to the group of structural nonlinearities and demonstrate how easily the nonlinear characteristic can be designed through appropriate combining of linear components.

1.1.4 Constraints

Unilateral constraints are another important example of structural nonlinearities. It is an important source of nonlinearities and it will be discussed extensively below. Here we give only one example (Fig. 1.7) showing the pendulum suspended near a rigid wall.

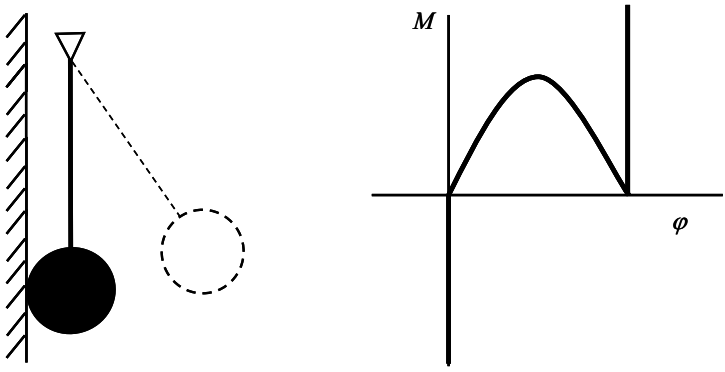

Fig. 1.7. The pendulum near the wall and its torque characteristic

This system is not simply nonlinear. It is much more complex. Some additional hypotheses are necessary in order to describe the collisions between the mass and the wall. In any case this system cannot be linearized at least as long as we remain in the framework of rigid body mechanics.

Kinematical constraints (not only unilateral) are an important source of nonlinearities. Although Newton's equations are linear with respect to coordinates and forces, the Lagrange's equations in generalized coordinates (which take constraints implicitly into account) are usually nonlinear. Consider the crank mechanism (Fig. 1.8) as an example.

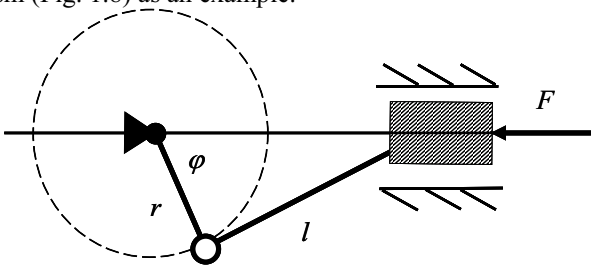

Fig. 1.8. Crank mechanism

Its position can be completely characterized by the angle φ between the crank and the horizontal axis. The position of the slider can be always expressed in terms of this angle:

$$x = r\cos\varphi + l\sqrt{1-\frac{r^2}{l^2}\sin^2\varphi} \tag{1.3}$$

This relationship is significantly nonlinear and lead to Lagrange's equation governing the mechanism:

$$\begin{aligned}
&T = \frac{1}{2}mr^2\dot{\varphi}^2 + \frac{1}{2}M\dot{x}^2 = \frac{1}{2}J(\varphi)\dot{\varphi}^2 \\
&J(\varphi) = mr^2 + Mr^2\sin^2\varphi\left(1+\frac{r^2}{l^2}\cos^2\varphi \Big/ \left(1-\frac{r^2}{l^2}\sin^2\varphi\right)\right) \\
&\delta A = F\delta x = Q\delta\varphi \Rightarrow \\
&Q = -r\sin\varphi\left(1+\frac{r}{l}\cos\varphi \Big/ \sqrt{1-\frac{r^2}{l^2}\sin^2\varphi}\right) \\
&\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\varphi}}\right) - \frac{\partial T}{\partial\varphi} = Q \Rightarrow J(\varphi)\ddot{\varphi} + \frac{1}{2}\frac{dJ}{d\varphi}\dot{\varphi}^2 = Q
\end{aligned} \tag{1.4}$$

Here mr^2 is the crank's inertia, M is the mass of the slider, T is the kinetic energy of the whole mechanism, Q is the generalized force obtained through the relationship for the virtual work.

This equation is totally nonlinear, but we would like to attract your attention to the second term on the left hand side. It depends on $\dot{\varphi}^2$. It is the consequence of the variable effective inertia $J(\varphi)$ due to kinematical coupling (1.3). This nonlinear dependency on the generalized velocities is usual for diverse mechanisms.

1.1.5 Nonlinearity of Friction

The last but not least source of nonlinearity which we are going to mention here is damping. Damping mechanisms are extremely complex and deeply connected with microscopic processes in materials, on their surfaces and in thin fluid films. Even the simplest models of viscous damping are nonlinear. Usually they can be represented in the following form:

$$\underline{R} = -f\left(|\underline{v}|\right)\frac{\underline{v}}{|\underline{v}|} \tag{1.5}$$

Here R is the damping force directed against the velocity. The function $f\left(|\underline{v}|\right)$ describes how the friction force depends on the magnitude of the velocity. The usual linear damping corresponds to $f\left(|\underline{v}|\right) = b|\underline{v}|$. This damping is extremely rare in applications. The only real case is the stationary flow in a long pipe. Nevertheless linear damping is very often used if the real damping mechanism is unknown, but some energy dissipation is necessary for the analysis.

More realistic is the power law $f(v) = b|\underline{v}|^{\alpha}$, $1 < \alpha \le 2$. It describes fluid damping at high Reynolds numbers. The case $\alpha = 2$ corresponds to the fully developed turbulent flow, which is typical for air.

Dry friction in the contact between two surfaces depends both on the relative velocity and on the normal force in the contact area: $f(v) = \mu N$. In the one dimensional case of dry friction the relationship (1.5) is usually written as follows:

$$R = -\mu N \operatorname{sgn}(v) \tag{1.6}$$

The friction coefficient μ is however not a constant. Even more, the formal relationship (1.6) is not a function. It is not defined for $v = 0$. This special case corresponds to the so called sticking and is usually described by a separate coefficient μ_s. The friction force during sticking cannot be calculated according to the equation (1.6). It is determined by the condition $v = 0$ as long as the calculated value doesn't exceed the maximal value:

$$|R| \le \mu_s N \tag{1.7}$$

Slipping starts as soon as the inequality (1.7) is broken if we suppose sticking.

The simplest Coulomb's friction law ($\mu = const$) is shown in Fig. 1.9 a). The friction law taking the sticking friction into account is shown in Fig. 1.9 b). Figure 1.9 c) shows the friction coefficient taking into account the negative slope of the force–velocity curve at small relative velocities. This decreasing friction coefficient was confirmed in numerous experiments for various friction partners and fluids (air, water, oil) between them. Figure 1.9 d) shows finally the regularized friction law used sometimes in numerical simulations. It doesn't take into account the possibility of sticking (the vertical line for $v = 0$ is replaced through a quasi viscous damping) and is applicable only to investigations in which sticking doesn't occur.

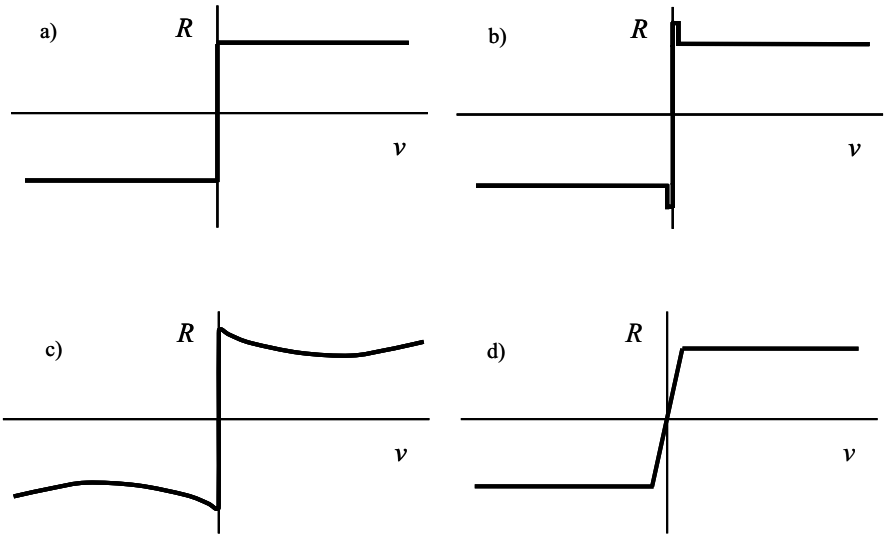

Fig. 1.9. Different idealization for dry friction

Additional information concerning dry friction and the corresponding references can be found in Chapter 2.

1.2 The Basic Ideas of the Perturbation Analysis

1.2.1 Variation of Free Constants and Systems in the Standard Form

The main idea of perturbation methods is to consider systems being close to an unperturbed one. It is supposed that the solutions of the unperturbed system are easy to find. In other words it is supposed that the unperturbed system can be integrated in a closed form.

Consider a system

$$\dot{z} = Z\left(z,t,\varepsilon\right) \tag{1.8}$$

Here $\varepsilon << 1$ is a small parameter. Consider the corresponding unperturbed system

$$\dot{z}_0 = Z\left(z_0,t,0\right) \tag{1.9}$$

We suppose that its general solution is known:

$$z_0 = f\left(t,C\right) \Leftrightarrow \frac{\partial f}{\partial t} = Z\left(f\left(t,C\right),t,0\right) \tag{1.10}$$

Here C is the vector of arbitrary constants. Taking C as a set of new variables, i.e. considering (1.10) as a transformation for the perturbed system (1.8), the following equation can be easily obtained:

$$\begin{aligned} \frac{\partial f}{\partial C}\dot{C} + \frac{\partial f}{\partial t} &= Z\left(f\left(t,C\right),t,\varepsilon\right) \\ &= Z\left(f\left(t,C\right),t,0\right) + \varepsilon \left.\frac{\partial Z}{\partial \varepsilon}\right|_{\varepsilon=0} + \ldots \end{aligned} \tag{1.11}$$

Here $\ldots$ stays for the terms $O\left(\varepsilon^2\right)$. Taking (1.10) into account and supposing the matrix $\left(\partial f/\partial C\right)$ to be not degenerated, i.e. $\det\left(\partial f/\partial C\right) \neq 0$ the following equation for the new variables C can be obtained:

$$\dot{C} = \varepsilon \left(\frac{\partial f}{\partial C}\right)^{-1} \left.\frac{\partial Z\left(f\left(t,C\right),t,\varepsilon\right)}{\partial \varepsilon}\right|_{\varepsilon=0} + \ldots \tag{1.12}$$

A system in form (1.12), i.e. a system which right hand side is multiplied by the small parameter, is called a "*system in the standard from*" for the averaging method. Usually it is written as follows:

$$\dot{x} = \varepsilon X\left(x,t,\varepsilon\right) \tag{1.13}$$

Here x is the n-dimensional vector of the state variables, X is the n-dimensional vector-function depending on the state variables, time and perhaps on the small parameter ε.

Actually the statement: "The system (1.13) is in the standard form because the small parameter stays as a factor in front of its right-hand side" is too simplified. The functions at the right hand side of (1.13) have to be in addition bounded and smooth, and the time average of the right hand sides must exist (see below). These additional conditions are not always easy to satisfy. Even if the unperturbed system is a linear excited and damped oscillator (1.14), but the excitation and damping are not small, it cannot be directly transformed to the standard form (cf. Chapter 6).

$$m\ddot{x} + \beta\dot{x} + cx = a\sin\omega t,\ a = O\left(1\right),\ \beta = O\left(1\right) \tag{1.14}$$

Nevertheless there are two large and very important classes of systems suitable for the perturbation analysis.

The quasi-conservative, especially quasi-linear systems belong to the first class [78]. Many classical examples for the perturbation analysis like Duffing's and van der Pol's equations [123] belong to this class of systems. Many other examples can be found in any classical book concerning perturbations (*cf.* [23, 79, 80, 99, 114]).

Systems with strong excitation, i.e. systems with dominating external and inertial forces, belong to the second class. Numerous examples of such systems can be found in [20, 21], some special problems are considered also in [119 – 124]. It is usual to write the governing equations for the problems of this class in a slightly different form:

$$\ddot{x} = F\left(x,\dot{x},t\right) + \omega\Phi\left(x,t,\tau\right),\qquad \tau = \omega t,\ \omega >> 1 \tag{1.15}$$

This form expresses better the fact that the term $\omega\Phi\left(x,t,\tau\right)$ containing the high-frequency excitation, is dominating here (because ω is the large parameter).

There are many different methods for the asymptotic analysis of perturbed dynamical systems. First of all, these methods differ according to the type of solutions they deal with. There are numerous methods considering only periodic solutions and their stability. Most of them are based on the ideas of Poincaré and Lyapunov.

Another group of methods considers also transient solutions of dynamical systems, i.e. these methods allow analyzing not only an infinitesimal vicinity of periodic solutions, but also their attraction area.

Three of these methods are most popular today. We are going to start with the standard averaging.

1.2.2 Standard Averaging as an Almost Identical Transformation

Averaging as a perturbation method was worked out in the middle of the 20th century first of all in the works of Russian mathematicians Bogoliubov and Mitropolskii [23, 24, 74, 75] and then developed by their colleagues both in Russia [7, 78, 82 – 84, 111, 133] and in the West [112 – 114].

Initial value problems in the standard form are investigated by the averaging.

$$\dot{x} = \varepsilon X\left(x,t,\varepsilon\right),\ \ x\left(0\right) = x_0 \tag{1.16}$$

Let us firstly introduce the time average of the function X :

$$\left\langle X\left(x,t,\varepsilon\right)\right\rangle_t = \lim_{T\to\infty}\frac{1}{T}\int_0^T X\left(x,t,\varepsilon\right)dt \tag{1.17}$$

The integration here has to be performed with respect to the explicit time (variables x are considered as constant parameters).

If the function X is periodic with respect to the explicit time t, the definition of the time average can be significantly simplified:

$$\begin{aligned} &\forall t : X\left(x,t+2\pi,\varepsilon\right) = X\left(x,t,\varepsilon\right) \Rightarrow \\ &\left\langle X\left(x,t,\varepsilon\right)\right\rangle_t = \frac{1}{2\pi}\int_0^{2\pi} X\left(x,t,\varepsilon\right)dt \end{aligned} \tag{1.18}$$

The main idea of the averaging method is not to try to solve the system (1.16), but to try to find another system, being simpler than the original one, which solutions are close to the solutions of the original system for a sufficiently long time interval. The simplification which can be achieved using averaging is to eliminate the independent variable t from the considered equations, i.e. to reduce the effective order of the system by one.

In order to do it formally (without mathematical proof) for the simplest periodic case, the following almost identical transformation can be applied:

$$x = \xi + \varepsilon u\left(\xi,t,\varepsilon\right) + O\left(\varepsilon^2\right) \tag{1.19}$$

It is very important to understand the sense of this transformation in order to comprehend the physical meaning of the method. It splits the solution to (1.16) into two parts – the large slowly varying part ξ, describing the evolution of the system, and the small fast oscillating part u, which is responsible for the oscillations of the solution around the slow component.

We require that the new variable ξ is governed by the autonomous equation

$$\dot{\xi} = \varepsilon \Xi\left(\xi, \varepsilon\right) + O\left(\varepsilon^2\right) \tag{1.20}$$

Both the unknown function $u\left(\xi, t, \varepsilon\right)$, which has to be a periodic function of time, and $\Xi\left(\xi, \varepsilon\right)$ have to be determined by the following procedure. Applying (1.19) and (1.20) to (1.16) the following equation can be obtained:

$$\varepsilon \Xi\left(\xi, \varepsilon\right) + \varepsilon \frac{\partial u}{\partial t} = \varepsilon X\left(\xi, t, \varepsilon\right) + O\left(\varepsilon^2\right) \tag{1.21}$$

Balancing the terms $O\left(\varepsilon\right)$ and requiring that u has to be a bounded periodic function, we obtain that this condition can be fulfilled only if Ξ is the time average of X:

$$\begin{aligned} &\Xi\left(\xi, \varepsilon\right) = \left\langle X\left(\xi, t, \varepsilon\right)\right\rangle_t \\ &u\left(\xi, t, \varepsilon\right) = \int_0^t \left(X\left(\xi, \vartheta, \varepsilon\right) - \Xi\left(\xi, \varepsilon\right)\right) d\vartheta + u_0\left(\xi, \varepsilon\right) \end{aligned} \tag{1.22}$$

It is usual to choose the free functions $u_0\left(\xi, \varepsilon\right)$ in order to guarantee that the time average of the functions u is equal to zero, i.e. $\left\langle u\left(\xi, t, \varepsilon\right)\right\rangle_t = 0$.

It is not the unique possible choice of the functions u_0. Another one is convenient if the equations (1.16) have the Hamiltonian form. Then the free functions can be chosen in order to guarantee that the averaged equations also have the Hamiltonian form.

Higher order approximations can be obtained in a similar way. For the second order approximation we apply the following transformation:

$$x = \xi + \varepsilon u_1\left(\xi, t, \varepsilon\right) + \varepsilon^2 u_2\left(\xi, t, \varepsilon\right) + \ldots \tag{1.23}$$

We require further that the new variable ξ is governed by an autonomous equation

$$\dot{\xi} = \varepsilon \Xi_1\left(\xi, \varepsilon\right) + \varepsilon^2 \Xi_2\left(\xi, \varepsilon\right) + \ldots \tag{1.24}$$

Balancing the terms $O\left(\varepsilon\right)$ we obtain as above:

$$\frac{\partial u_1}{\partial t} = X(\xi,t,\varepsilon) - \Xi_1(\xi,\varepsilon) \Rightarrow$$
$$\Xi_1(\xi,\varepsilon) = \langle X(\xi,t,\varepsilon)\rangle_t, \quad u_1 = \int\left[X(\xi,t,\varepsilon) - \Xi_1(\xi,\varepsilon)\right]dt, \tag{1.25}$$
$$\langle u_1\rangle_t = 0$$

Balancing the terms $O(\varepsilon^2)$ we obtain:

$$\frac{\partial u_2}{\partial t} = \left.\frac{\partial X}{\partial x}\right|_{x=\xi} u_1 - \Xi_2 - \frac{\partial u_1}{\partial \xi}\Xi_1 \Rightarrow$$
$$\Xi_2 = \left\langle \left.\frac{\partial X}{\partial x}\right|_{x=\xi} u_1 - \frac{\partial u_1}{\partial \xi}\Xi_1 \right\rangle_t = \left\langle \left.\frac{\partial X}{\partial x}\right|_{x=\xi} u_1 \right\rangle_t \tag{1.26}$$

Finally the equation of the second order approximation is

$$\dot{\xi} = \varepsilon\langle X(\xi,t,\varepsilon)\rangle + \varepsilon^2 \left\langle \left.\frac{\partial X}{\partial x}\right|_{x=\xi} u_1 \right\rangle_t + O(\varepsilon^3) \tag{1.27}$$

The procedure above is purely formal, because it does not explain if we can shorten the equations for ξ neglecting the small terms $O(\varepsilon^2)$ or $O(\varepsilon^3)$ in the equations (1.20) and (1.27) respectively. It also doesn't explain why and for how long a time the solutions of the original system (1.16) and those of the shortened averaged systems (1.20) or (1.27) are close to each other.

Answers to these questions were given by Bogoliubov in his first **Theorem**.

Consider the system (1.16) *and assume*

1. X *is a measurable with respect to* t *vector-function.*
2. *It is bounded and satisfies the Lipschitz-condition with respect to the vector-argumen* x.

$$\|X(x,t,\varepsilon)\| \le M$$
$$\|X(x_1,t,\varepsilon) - X(x_2,t,\varepsilon)\| \le \lambda\|x_1 - x_2\| \tag{1.28}$$

3. *The time average of the function* X *exists uniformly with respect to* x.

Consider the averaged system satisfying the same initial conditions

$$\dot{\xi} = \varepsilon\Xi(\xi,\varepsilon), \quad \xi(0) = x_0 \tag{1.29}$$

Under these conditions the mistake by using the system (1.29) *with functions* Ξ *determined by the relationships* (1.22) *instead of the original one has the magnitude order of the small parameter* ε *for the asymptotically long time interval* $t = O(1/\varepsilon)$.

The proof of this theorem is not very complex. Readers interested in the mathematical background can find it in Appendix I.

If the averaged system (1.29) has an asymptotically stable singular point in the linear approximation and the function X is continuously differentiable with respect to x then the original system (1.16) has a solution which remains in the vicinity of this point for infinite time.

This is the contents of the second Bogoliubov's **Theorem**. Its mathematically correct formulation and proof can be found in Appendix II.

1.2.3 Method of Multiple Scales

"*Multiple scales*" is surely the most popular method for the asymptotic analysis of dynamical systems in the west. It differs slightly in form from the averaging method, but is substantially very close to it. The multiple scales technique is very simple and straight forward in its formulation, its logic is easy to understand and to learn and it is excellently explained in [79, 80]. Thanks to numerous works by A.H. Nayfey this method is successfully used for analyzing both ordinary and partial differential equations, although its accuracy is not always mathematically guarantied. Correct use of multiple scales in these cases requires deep physical understanding of the problem.

The basic logic of the method of multiple scales can be easily illustrated by considering a system in the standard form (1.16). The first step of the solution is to convert to two independent variables $\theta = t$ and $\tau = \varepsilon t$, supposing that $x = \varphi(\theta, \tau)$, i.e. to convert from the system of ordinary differential equations (1.16) to the following system with partial derivatives:

$$\frac{\partial \varphi}{\partial \theta} + \varepsilon \frac{\partial \varphi}{\partial \tau} = \varepsilon X\left(\varphi(\theta, \tau), \varepsilon\right) \tag{1.30}$$

The relationship between (1.16) and (1.30) is determined by the condition that, if $\varphi(\theta, \tau)$ is a solution to (1.30), then $x = \varphi(t, \varepsilon t)$ is a solution to (1.16). In other words, the system (1.30) is more general then the original equations (1.16) and any solution to (1.30) taken along the straight line $\theta = t,\ \tau = \varepsilon t$ satisfies the equations (1.16).

We require $\varphi(\theta, \tau)$ to be a 2π -periodic function of τ and try to find φ as a formal asymptotic expansion in terms of ε:

$$\varphi(\theta,\tau)=\psi_0(\theta,\tau)+\varepsilon\psi_1(\theta,\tau)+\ldots \tag{1.31}$$

All the functions here have to be bounded functions of the fast time θ. Substituting this expression into equation (1.30) and balancing the terms with equal powers of ε the following relationships can be obtained:

$$\begin{aligned}\varepsilon^0:&\quad \frac{\partial\psi_0}{\partial\theta}=0\\ \varepsilon^1:&\quad \frac{\partial\psi_0}{\partial\tau}+\frac{\partial\psi_1}{\partial\theta}=X\left(\psi_0(\theta,\tau),\theta,\varepsilon\right)\end{aligned} \tag{1.32}$$

The first of these equations means that ψ_0 depends only on the slow time τ:

$$\psi_0=\xi(\tau) \tag{1.33}$$

Substituting (1.33) into the second relationship from (1.32) a simple equation for $\psi_1(\theta,\tau)$ can be obtained:

$$\frac{\partial\psi_1}{\partial t}=X\left(\xi(\tau),t,\varepsilon\right)-\frac{d\xi}{d\tau} \tag{1.34}$$

The function $\psi_1(\theta,\tau)$ has to be a bounded function of θ, i.e. its derivative can contain only oscillating components. It is possible if the function $\frac{d\xi}{d\tau}$ annihilates the constant component of X. It means

$$\frac{d\xi}{d\tau}=\left\langle X\left(\xi,\theta,\varepsilon\right)\right\rangle_\theta \tag{1.35}$$

Here the average is calculated with respect to the fast time θ. Returning back to the straight line $\theta=t,\ \tau=\varepsilon t$ we find that the slow component of the solution is governed by the following equation:

$$\frac{d\xi}{dt}=\varepsilon\left\langle X\left(\xi,t,\varepsilon\right)\right\rangle_t \tag{1.36}$$

The fast oscillating small correction ψ_1 can be calculated as follows:

$$\psi_1=\int_0^t\left(X\left(\xi,\vartheta,\varepsilon\right)-\left\langle X\left(\xi,\vartheta,\varepsilon\right)\right\rangle_\vartheta\right)d\vartheta+\psi_1^0\left(\xi,\varepsilon\right) \tag{1.37}$$

Comparing (1.31) and (1.35) - (1.37) with the corresponding relationships from the previous subsection describing the averaging method (1.19) and (1.22), (1.29) it is easy to notice that they are identical ($\psi_1 \equiv u$). Considering the higher order terms in the expansion (1.31), the higher order approximations to (1.16) can be obtained by the multiple scales technique. They are the same as those obtained by the averaging method.

1.2.4 Direct Separation of Motions

The method of the *direct separation of motions* was originated by Kapitsa [55, 56], who used it for analyzing a pendulum with a vibrating suspension point. The general formulation of this method was given later by Blekhman [20], who described it as the most effective method of vibrational mechanics. It was also used to solve numerous dynamical problems both in Russia and, in the last 10 years, also in the west [16 – 22, 53, 73, 119 – 124].

Direct separation of motions was formulated originally for systems of second order differential equations, but it can be easily reformulated as follows.

Consider a system of ordinary differential equations:

$$\dot{x} = F\left(x, \varepsilon t\right) + \Phi\left(x, \varepsilon t, t\right) \tag{1.38}$$

The basic idea of the direct separation of motions is to consider only the solutions, which are a superposition of slow evolution and fast oscillations. The object of main interest is the slow component:

$$\begin{aligned} & x\left(t\right) = \xi\left(\tau\right) + \psi\left(\tau, t\right) \\ & \tau = \varepsilon t; \ \left\langle \psi\left(\tau, t\right)\right\rangle_t = \frac{1}{2\pi}\int_0^{2\pi} \psi\left(\tau, t\right) dt = 0 \end{aligned} \tag{1.39}$$

The next step is to go over from the system of n differential equations (1.38) to a system of $2n$ integral-differential equations:

$$\begin{aligned} \dot{\xi} = {} & F\left(\xi, \tau\right) + \left\langle F\left(\xi + \psi\left(t, \tau\right), \tau\right) - F\left(\xi, \tau\right)\right\rangle_t \\ & + \left\langle \Phi\left(\xi + \psi\left(t, \tau\right), \tau, t\right)\right\rangle_t \\ \dot{\psi} = {} & F\left(\xi + \psi\left(t, \tau\right), \tau\right) - F\left(\xi, \tau\right) \\ & - \left\langle F\left(\xi + \psi\left(t, \tau\right), \tau\right) - F\left(\xi, \tau\right)\right\rangle_t \\ & + \Phi\left(\xi + \psi\left(t, \tau\right), \tau, t\right) - \left\langle \Phi\left(\xi + \psi\left(t, \tau\right), \tau, t\right)\right\rangle_t \end{aligned} \tag{1.40}$$

The relationship between systems (1.38) and (1.40) is as follows: if a pair (ξ,ψ) is a solution to (1.40), then $x(t)$ determined according to (1.39) is automatically a solution to (1.38). It means the system (1.40) is more general than the original one. This system is not only more general, but it is at first sight also more complex. Nevertheless in many important cases it is easy to solve with the assumption that the variable ξ in the second equation (1.40) is constant.

The system in the standard form (1.16) can be considered as an example. In this case

$$F(x,\varepsilon t)=0;\quad \Phi(x,\varepsilon t,t)=\varepsilon X(x,t) \tag{1.41}$$

Substituting (1.41) into (1.40) the following equations can be easily obtained:

$$\begin{aligned}\dot{\xi}&=\varepsilon\left\langle X(\xi+\psi,t)\right\rangle_t\\ \dot{\psi}&=\varepsilon X(\xi+\psi,t)-\varepsilon\left\langle X(\xi+\psi,t)\right\rangle_t\end{aligned} \tag{1.42}$$

It is natural to solve the second equation of the system (1.42) asymptotically:

$$\psi=\psi_0+\varepsilon\psi_1+\ldots \tag{1.43}$$

Inserting this expression into the second equation (1.42) and balancing terms with the equal powers of the small parameter, it is easy to see, that

$$\begin{aligned}\psi_0&=0\\ \dot{\xi}&=\varepsilon\left\langle X(\xi,t)\right\rangle\\ \dot{\psi}_1&=X(\xi,t)-\left\langle X(\xi,t)\right\rangle\end{aligned} \tag{1.44}$$

Equations (1.44) don't differ from the equations of the first order approximation (1.20), (1.22) or (1.36), (1.37).

Unfortunately neither the method of multiple scales nor the method of the direct separation of motions has a mathematical proof differing from that for the standard averaging.

1.2.5 Relationship between These Methods

All the considered methods are very useful and efficient in the analysis of oscillating systems. All of them applied to the system in standard form give the same result. (It is actually the necessary condition for such a procedure to be called a method.) So the choice of one of them in any special case is connected mainly with personal preference. From the authors point of view the multiple scales, and especially the direct separation of motions, are slightly easier for the practical use compared to the standard averaging method. Their main advantage is the straight

forward algorithm used to solve the problem, which does not require to initially transform the system to the standard form. This transformation may be sometimes rather difficult.

But this statement is correct only for systems which are in or can be transformed to the standard form. The situation becomes much more interesting if it is impossible to transform a system to the standard form or one of the conditions (1.28) or (1.29) is not fulfilled. In such a case it is possible to try any of the described methods. The problem is how to make sure that the obtained results are correct. The main advantage of the averaging method becomes clear in these cases. There is a clear way, based on the Gronwall's lemma, to prove the accuracy of the averaging procedure. Thus the method contains the instrument to generalize itself. This situation enables researchers move away from pure physical intuition (being the most effective in many cases) and to take the path of rigorous mathematical analysis.

1.3 Examples of Elementary Nonlinear Problems Solved by Standard Averaging

Here we give three academic examples of simple nonlinear problems, which can be solved by asymptotic methods.

1.3.1 Instability and Self Excited Oscillations in the Van Der Pol's Equation

Consider the simplest equation with self excitation:

$$\ddot{x} - \varepsilon\left(1 - x^2\right)\dot{x} + x = 0, \quad 0 < \varepsilon << 1 \tag{1.45}$$

Damping in this system is negative for sufficiently small values of x. Thus the trivial equilibrium $x = 0$ is unstable. For sufficiently large values of x the term $1 - x^2$ is negative and the effective damping gets positive. The objective is to find the limit cycle enveloping the unstable equilibrium.

The unperturbed system to (1.45) is ($\varepsilon = 0$):

$$\ddot{x}_0 + x_0 = 0 \tag{1.46}$$

It is a harmonic linear oscillator. Its general solution is well known:

$$x_0 = A\sin\left(t + \theta\right);\ \dot{x}_0 = A\cos\left(t + \theta\right) \tag{1.47}$$

Here A and θ are the free integration constants. This solution can be used as the transformation in the perturbed equation (1.45):

$$x = A\sin\left(t+\theta\right);\ \dot{x} = A\cos\left(t+\theta\right) \tag{1.48}$$

This transformation is named after van der Pol [99], who suggested it in 1926. The new variables A and θ are governed by the following equations:

$$\begin{aligned}\dot{A} &= \varepsilon A\cos^2\left(t+\theta\right)\left(1-A^2\sin^2\left(t+\theta\right)\right)\\ \dot{\theta} &= -\varepsilon\sin\left(t+\theta\right)\cos\left(t+\theta\right)\left(1-A^2\sin^2\left(t+\theta\right)\right)\end{aligned} \tag{1.49}$$

This system has the standard form for averaging. It can be averaged with respect to t. The equations of the first order approximation are as follows:

$$\begin{aligned}\dot{A}_1 &= \frac{1}{2}\varepsilon\left(A_1 - \frac{1}{4}A_1^3\right)\\ \dot{\theta}_1 &= 0\end{aligned} \tag{1.50}$$

Index 1 indicates here the first order approximation. These equations have two stationary solutions with respect to the amplitude:

$$A_{10} = 0;\ A_{11} = 2 \tag{1.51}$$

The first one is unstable, the second one stable. It describes the stable limit cycle in the original phase plane.

In this case the averaged system can be integrated completely. The result is

$$\begin{aligned}A_1 &= \frac{2}{\sqrt{1+e^{-\varepsilon t-t_0}}}\\ \theta_1 &= \theta_0\end{aligned} \tag{1.52}$$

The constants t_0 and θ_0 are determined by the initial conditions. Returning back to the original variables we obtain the following first order approximation:

$$x_1 = \frac{2\sin\left(t+\theta_0\right)}{\sqrt{1+e^{-\varepsilon t-t_0}}};\ \dot{x}_1 = \frac{2\cos\left(t+\theta_0\right)}{\sqrt{1+e^{-\varepsilon t-t_0}}} \tag{1.53}$$

The stable limit cycle is

$$x_{1\infty} = 2\sin\left(t+\theta_0\right);\ \dot{x}_{1\infty} = 2\cos\left(t+\theta_0\right) \tag{1.54}$$

Examples of the transient solutions with $x_0 < 2$ and $x_0 > 2$ are shown in Fig. 1.10 for $\varepsilon = 0.3$.

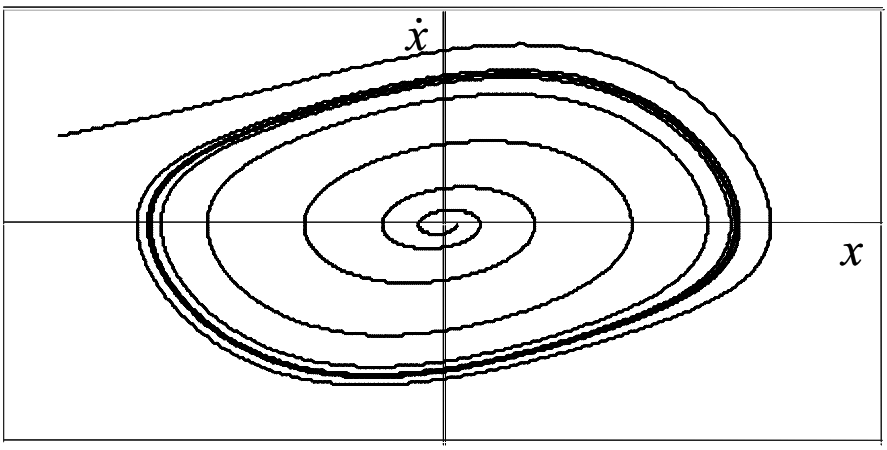

Fig. 1.10. Transient solutions of the van der Pol's equation converging to the stable limit cycle

1.3.2 The Main Resonance in a System with a Small Cubic Nonlinearity

Consider the resonance problem in a system with a small cubic nonlinearity:

$$\ddot{x} + 2\beta\dot{x} + x + \varepsilon x^3 = a\cos\omega t, \quad \varepsilon << 1,\ a = O(\varepsilon),\ \beta = O(\varepsilon) \tag{1.55}$$

We assume that the amplitude of the external excitation, damping and the nonlinear term are small.

The unperturbed system corresponding to (1.55) is the same as in the previous example. So we can apply the van der Pol's transformation, which we are going to use now in a slightly different form:

$$x = A\sin\varphi;\ \dot{x} = A\cos\varphi \tag{1.56}$$

Here φ is the almost uniformly rotating phase. For the unperturbed system $\varphi = t + const$. The new variables are governed by the following equations:

$$\begin{aligned} \dot{A} &= -2\beta A\cos^2\varphi + a\cos\omega t\cos\varphi - \varepsilon A^3\sin^3\varphi\cos\varphi \\ \dot{\varphi} &= 1 + 2\beta\cos\varphi\sin\varphi - \frac{a}{A}\cos\omega t\sin\varphi + \varepsilon A^2\sin^4\varphi \end{aligned} \tag{1.57}$$

We are going to investigate the main resonance, i.e. the frequency of the external excitation ω is close to the natural frequency of the unperturbed system, which in this case is equal to one. We can assume that the frequency delay is small, i.e. it has the same magnitude order as the small parameter:

$$\omega = 1 + \delta,\ \delta = O(\varepsilon) \tag{1.58}$$

Then equations (1.57) can be rewritten as follows:

$$\begin{aligned} \dot{A} &= -2\beta A\cos^2\varphi + a\cos\psi\cos\varphi - \varepsilon A^3\sin^3\varphi\cos\varphi \\ \dot{\varphi} &= 1 + 2\beta\cos\varphi\sin\varphi - \frac{a}{A}\cos\psi\sin\varphi + \varepsilon A^2\sin^4\varphi \\ \dot{\psi} &= 1 + \delta \end{aligned} \tag{1.59}$$

Here we have introduced the second phase $\psi = \omega t$. The two phases φ and ψ have almost the same rotation speeds. Thus it is sensible to introduce the phase difference between them:

$$\theta = \varphi - \psi \tag{1.60}$$

Now we can rewrite equations (1.59) replacing φ through $\psi + \theta$:

$$\begin{aligned} \dot{A} &= -2\beta A\cos^2(\psi+\theta) + a\cos\psi\cos(\psi+\theta) \\ &\quad -\varepsilon A^3\sin^3(\psi+\theta)\cos(\psi+\theta) \\ \dot{\theta} &= -\delta + 2\beta\cos(\psi+\theta)\sin(\psi+\theta) - \frac{a}{A}\cos\psi\sin(\psi+\theta) \\ &\quad +\varepsilon A^2\sin^4(\psi+\theta) \\ \dot{\psi} &= 1 + \delta \end{aligned} \tag{1.61}$$

This system contains two slow variables A and θ. The fast phase ψ can be considered as the independent variable. Thus the system (1.61) has the standard form and it can be averaged with respect to ψ. Equations of the first order approximation are:

$$\begin{aligned} \dot{A}_1 &= -\beta A_1 + \frac{1}{2}a\cos\theta_1 \\ \dot{\theta}_1 &= -\delta - \frac{a}{2A_1}\sin\theta_1 + \frac{3}{8}\varepsilon A_1^2 \end{aligned} \tag{1.62}$$

If we are interested in the stationary resonant solutions, we can determine their amplitude from the following equations (the right hand sides of the equations (1.62) must be equal to zero):

$$a\cos\theta_{10} = 2\beta A_{10}$$
$$a\sin\theta_{10} = -2\delta A_{10} + \frac{3}{4}\varepsilon {A_{10}}^3 \tag{1.63}$$

The stationary amplitude is determined by the algebraic equation

$$4\beta^2 {A_{10}}^2 + {A_{10}}^2\left(2\delta - \frac{3}{4}\varepsilon {A_{10}}^2\right)^2 = a^2 \tag{1.64}$$

This equation can have up to three positive solutions. The corresponding resonance curve is shown in Fig. 1.11.

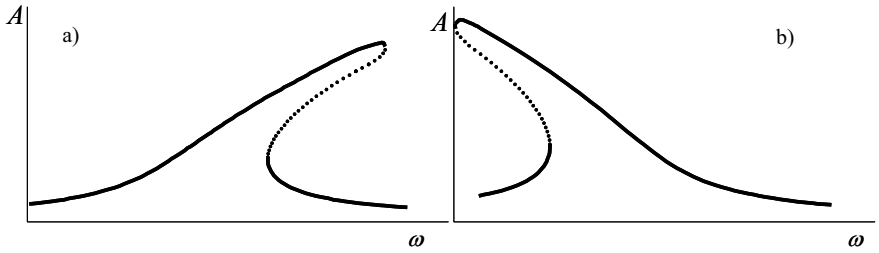

Fig. 1.11. Stationary resonant amplitudes; a) the hard nonlinearity $\varepsilon > 0$; b) the soft nonlinearity $\varepsilon < 0$

It is easy to show that the upper and the lower branches of these curves are stable (solid line) and the middle branch is always unstable (dotted line).

An important indicator of the resonance in a nonlinear system is the difference between the stationary amplitudes if one increases (or decreases) the excitation frequency. Consider the hard nonlinearity for example. If we increase slowly the excitation's frequency, the amplitude follows the upper curve as long as possible. Then it falls down abruptly and follows the lower curve. If we decrease the frequency beginning with a high level, which is over the resonance, then the amplitude follows the lower curve as long as possible and then jumps up to the upper curve. Thus the amplitude in the first case is larger than in the second case.

If we consider the soft nonlinearity then everything is the other way round.

Further discussions of this standard problem can be found in the textbooks [80, 120].

1.3.3 Secondary Resonances in the System with Cubic Nonlinearity and Strong Excitation

The considered resonance at $\omega \approx 1$ is the strongest but not the only one in the system with cubic nonlinearity. Let us assume the amplitude of the excitation is not small and investigate if additional resonances are possible in that system:

$$\ddot{x}+2\beta\dot{x}+x+\varepsilon x^3=a\cos\omega t,\quad \varepsilon<<1,\ a=O(1),\ \beta=O(\varepsilon) \tag{1.65}$$

The unperturbed system corresponding to (1.65) is ($\varepsilon=0,\ \beta=0$):

$$\ddot{x}_0+x_0=a\cos\omega t \tag{1.66}$$

Its general solution can be easily obtained if we exclude the main resonance from the further analysis ($\omega\neq 1$):

$$\begin{aligned} x_0&=A\sin(t+\alpha)+\frac{a}{1-\omega^2}\cos\omega t\\ \dot{x}_0&=A\cos(t+\alpha)-\frac{a\omega}{1-\omega^2}\sin\omega t \end{aligned} \tag{1.67}$$

A and α are the free constants here. We apply the modified van der Pol's transformation based on the solution (1.67) in order to investigate the perturbed system:

$$\begin{aligned} &x=A\sin\varphi+2\lambda\cos\omega t;\quad \dot{x}=A\cos\varphi-2\lambda\omega\sin\omega t\\ &\lambda=\frac{a}{2(1-\omega^2)}=O(1) \end{aligned} \tag{1.68}$$

The new variables A and φ are governed by the following equations:

$$\begin{aligned} \dot{A}&=-2\beta\cos\varphi(A\cos\varphi-2\lambda\omega\sin\psi)\\ &\quad -\varepsilon\cos\varphi(A\sin\varphi+2\lambda\cos\psi)^3\\ \dot{\varphi}&=1+\frac{2\beta}{A}\sin\varphi(A\cos\varphi-2\lambda\omega\sin\psi)\\ &\quad +\frac{\varepsilon}{A}\sin\varphi(A\sin\varphi+2\lambda\cos\psi)^3\\ \dot{\psi}&=\omega \end{aligned} \tag{1.69}$$

What is a resonance in such a system? Resonance is such a combination of parameters that the time average of the right hand sides gets discontinuous. The basic idea behind this definition is as follows. Consider a product of two trigonometric functions $\sin\omega_1 t\sin\omega_2 t$. Its time average is always equal to zero except one parameter combination $\omega_1=\omega_2$. Then the time average is equal to $1/2$. Thus this parameter combination corresponds to the resonance.

Let us investigate how this definition works for the system (1.69). These equations can be transformed to more convenient form if we use trigonometric identities:

$$\begin{aligned}
&\cos^3 x = \tfrac{3}{4}\cos x + \tfrac{1}{4}\cos 3x \\
&\sin^3 x = \tfrac{3}{4}\sin x - \tfrac{1}{4}\sin 3x \\
&\cos x \sin^2 x = \tfrac{1}{4}\cos x - \tfrac{1}{4}\cos 3x
\end{aligned} \tag{1.70}$$

Applying (1.70) to (1.69) one obtains:

$$\begin{aligned}
\dot{A} &= -2\beta A\cos^2\varphi + 4\beta\lambda\omega\cos\varphi\sin\psi - \varepsilon A^3\cos\varphi\sin^3\varphi \\
&\quad -\frac{3}{2}\varepsilon A^2\lambda\left(\cos\varphi - \cos 3\varphi\right)\cos\psi \\
&\quad -3\varepsilon A\lambda^2\sin 2\varphi\left(1+\cos 2\psi\right) \\
&\quad -2\varepsilon\lambda^3\cos\varphi\left(3\cos\psi + \cos 3\psi\right) \\
\dot{\varphi} &= 1 + \beta\sin 2\varphi - \frac{4\beta\lambda\omega}{A}\sin\varphi\sin\psi + \varepsilon A^2\sin^4\varphi \\
&\quad +\frac{3}{2}\varepsilon A\lambda\left(3\sin\varphi - \sin 3\varphi\right)\cos\psi + \\
&\quad +3\varepsilon\lambda^2\left(1-\cos 2\varphi\right)\left(1+\cos 2\psi\right) \\
&\quad +2\varepsilon\frac{\lambda^3}{A}\sin\varphi\left(3\cos\psi + \cos 3\psi\right) \\
\dot{\psi} &= \omega
\end{aligned} \tag{1.71}$$

Which terms in these equations can produce a discontinuous average? In order to see that we can replace φ through t and ψ through ωt. Then it becomes obvious that the main resonance corresponds to the parameter constellation $\omega = 1$. This case was investigated in the previous subsection. There are however two further parameter constellations producing discontinuous terms:

$$\begin{aligned}
&\omega = 3 \rightarrow \cos 3\varphi\cos\psi \\
&\omega = \frac{1}{3} \rightarrow \cos\varphi\cos 3\psi
\end{aligned} \tag{1.72}$$

These frequencies correspond to the secondary resonances in our system.

Let us consider the first case $\omega = 3$. Here the natural frequency of the linearized system is smaller than the frequency of the external excitation. The corresponding resonance is called "*sub-harmonic*".

In the second case $\omega = 1/3$ and the natural frequency is larger than the frequency of the external excitation. The corresponding resonance is called "*superharmonic*".

Approximate predictions for the stationary amplitudes in these cases can be obtained similarly to the previous subsection. The corresponding results can be found in [80, 120].

1.4 Axiomatic Theory of Collisions

Great attention in the present book is paid to discontinuous systems, first of all to systems with friction and collisions. Theory describing collisions of solid bodies doesn't belong to standard courses in technical mechanics for engineers. Thus we give here a short overview of the classical stereo mechanical theory of collisions.

Theory describing collisions of solid bodies is very complex and not really completed yet. Different approaches based on formal axioms are developed in [4, 77, 109, 135]. Physical modeling based on the wave propagation in the colliding bodies is discussed in [52, 140]. Quasi static approach based on the Hertz' contact model was developed in [42, 54]. An effective approach based on the visco-elastic contact elements inserted between the colliding bodies is discussed in [26, 42, 51, 52]. An overview of different theories and discussion of their applicability can be found in [26, 52]. Here we discuss shortly the classical stereo-mechanical approach.

1.4.1 Impulsive Motion of the Point Mass

The usual assumption of the axiomatic impact theory is that the duration of a collision is short and thus the change of the positions during this time interval is negligible. The change of the velocities of the colliding bodies is on the contrary large (comparable with the velocities just before the collision). It is possible if the contact forces acting during the collision are very large. Infinite forces are necessary for the finite change of the velocity during an infinitesimally small time interval. It is useful to replace the contact forces by their integral characteristic, which is called "*impulse*":

$$\underline{I} = \int_{t_-}^{t_+} \underline{F}(t)\, dt \tag{1.73}$$

Here t_- corresponds to the beginning of the collision and t_+ corresponds to its end, vector $\underline{F}$ is the contact force and vector $\underline{I}$ is its impulse.

Consider a point mass m subjected to a "normal" force $\underline{\underline{Q}}(t)$ alongside with the contact force $\underline{F}$. Change in position $\underline{r}$ and velocity $\underline{v}$ of the mass during impact can be calculated as follows:

$$\begin{aligned} m\underline{v}(t_+) - m\underline{v}(t_-) &= \underline{I} + \int_{t_-}^{t_+} \underline{\underline{Q}}(t)dt \\ \underline{r}(t_+) - \underline{r}(t_-) &= \int_{t_-}^{t_+} \underline{v}(t)dt \end{aligned} \tag{1.74}$$

The integral terms can be neglected for the infinitesimally short collisions. Thus we obtain the relationship for the velocity change during the collision:

$$\begin{aligned} m\Delta\underline{v} &= m\left(\underline{v}^+ - \underline{v}^-\right) = \underline{I} \\ \underline{v}^+ &= \underline{v}(t_+); \underline{v}^- = \underline{v}(t_-) \end{aligned} \tag{1.75}$$

1.4.2 Impulsive Motion of a System of Point Masses

These relationships can be easily generalized for a system of point masses $m_i, i = 1, 2, \ldots, n$ subjected to external forces $\underline{\underline{Q}}_i$ and interaction forces $\underline{R}_{ij}$ alongside the impulsive contact forces $\underline{F}_i$. The interaction forces satisfy the Newton's law (action is equal to reaction):

$$\underline{R}_{ij} = -\underline{R}_{ji}; \ \underline{R}_{ii} = 0 \tag{1.76}$$

Equations of motion for this system are:

$$\begin{cases} m_i \dot{\underline{v}}_i = \underline{\underline{Q}}_i + \underline{F}_i + \sum_{j=1}^{n} \underline{R}_{ij} \\ m_i \dot{\underline{r}}_i = \underline{v}_i \end{cases}, \ i = 1, 2, \ldots, n \tag{1.77}$$

Integrating these equations during a collision and neglecting the finite terms, we obtain the equations governing the change of the velocities due to the collision:

$$m_i \Delta \underline{v}_i = \underline{I}_i + \sum_{j=1}^{n} \underline{I}^h_{ij},\ i = 1,2,\ldots,n$$
$$\underline{I}^h_{ij} = \int_{t_-}^{t_+} \underline{R}^h_{ij}(t)\,dt;\ \underline{I}^h_{ij} = -\underline{I}^h_{ji};\ \underline{I}^h_{ii} = 0 \tag{1.78}$$

The terms $\underline{I}^h_{ij}$ are the impulses of the reaction forces. It is sensible to distinguish between the hard couplings (due to kinematical constraints) and soft couplings (like springs or dampers). The impulses in the soft couplings are equal to zero, the impulses in the hard couplings are finite. We will use an index h for these couplings.

All the classical theorems of dynamics can be reformulated for the impulsive motion. We can introduce the mass of the whole system, position of the gravity center and its velocity as follows:

$$M = \sum_{i=1}^{n} m_i;\ \underline{r}_G = \frac{1}{M}\sum_{i=1}^{n} m_i \underline{r}_i;\ \underline{v}_G = \frac{1}{M}\sum_{i=1}^{n} m_i \underline{v}_i \tag{1.79}$$

The changes of these values due to the collision are:

$$M \Delta \underline{v}_G = \sum_{i=1}^{n} \underline{I}_i,\ \sum_{i=1}^{n}\sum_{j=1}^{n} \underline{I}^h_{ij} = 0 \tag{1.80}$$

Similarly we can calculate the impulsive change of the kinetic torque with respect to any fixed point:

$$\underline{K}_O = \sum_{i=1}^{n} m_i \left(\underline{r}_i - \underline{r}_O\right) \times \underline{v}_i$$
$$\Delta \underline{K}_O = \sum_{i=1}^{n} m_i \left(\underline{r}_i - \underline{r}_O\right) \times \Delta \underline{v}_i = \sum_{i=1}^{n} \left(\underline{r}_i - \underline{r}_O\right) \times \underline{I}_i \tag{1.81}$$

Here we use the assumption that

$$\left(\underline{r}_i - \underline{r}_j\right) \times \underline{I}^h_{ij} = 0 \tag{1.82}$$

It means that the interaction between two point masses is collinear to the line connecting them (this is an independent axiom in mechanics).

Finally we can calculate the impulsive change of the kinetic energy.

$$\Delta T = \frac{1}{2}\sum_{i=1}^{n} m_i \left(\left|\underline{v}_i^+\right|^2 - \left|\underline{v}_i^-\right|^2 \right) = \frac{1}{2}\sum_{i=1}^{n} m_i \left(\underline{v}_i^+ - \underline{v}_i^- \right) \cdot \left(\underline{v}_i^+ + \underline{v}_i^- \right)$$
$$= \frac{1}{2}\sum_{i=1}^{n} \underline{I}_i \cdot \left(\underline{v}_i^+ + \underline{v}_i^- \right) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \underline{I}_{ij}^h \cdot \left(\underline{v}_i^+ + \underline{v}_i^- \right) \tag{1.83}$$

Here we have used the relationship (1.78). The hard couplings link two point masses rigidly. It means the distance between these masses does not change during the collision. Hence

$$\left(\underline{r}_i - \underline{r}_j \right) \cdot \left(\underline{v}_i^{\pm} - \underline{v}_j^{\pm} \right) = 0 \tag{1.84}$$

Index $\pm$ here means that it doesn't matter if we take the velocities before or after the collision. But according to (1.82) the internal impulses of the hard couplings are collinear with the vector $\left(\underline{r}_i - \underline{r}_j \right)$. Hence we obtain the relationships:

$$\underline{I}_{ij}^h \cdot \left(\underline{v}_i^+ - \underline{v}_j^+ \right) = 0;\ \underline{I}_{ij}^h \cdot \left(\underline{v}_i^- - \underline{v}_j^- \right) = 0 \tag{1.85}$$

Applying these relationships to the last term in (1.83) and taking the skew symmetry of the matrix $\underline{I}_{ij}^h$ into account (*cf.* (1.78)) we can see that this term vanishes. So we finally obtain the Kelvin's formula [4] for the change of the kinetic energy due to the collision:

$$\Delta T = \frac{1}{2}\sum_{i=1}^{n} \underline{I}_i \cdot \left(\underline{v}_i^+ + \underline{v}_i^- \right) \tag{1.86}$$

1.4.3 Impulsive Motion of a Rigid Body

The obtained results can be easily generalized in order to describe the impulsive motion of a rigid body. Let us introduce two frames (see Fig. 1.12).

The frame $O'x'y'z'$ doesn't move. The frame $Oxyz$ is connected with the rigid body V, which does not move relative to it. Velocity of an arbitrary point A is

$$\underline{v}_A = \underline{v}_O + \underline{\Omega} \times \underline{r}_A \tag{1.87}$$

Here $\underline{v}_O$ is the velocity of the pole O, $\underline{\Omega}$ is the angular velocity of the body. It is sensible to place the origin O into the gravity center of the body. Then according to (1.80)

$$M\Delta\underline{v}_G = \sum_{i=1}^{n} \underline{I}_i \tag{1.88}$$

Here M is the mass of the body, $\underline{I}_i$ are the impulses of the contact forces, which are applied in points A_i.

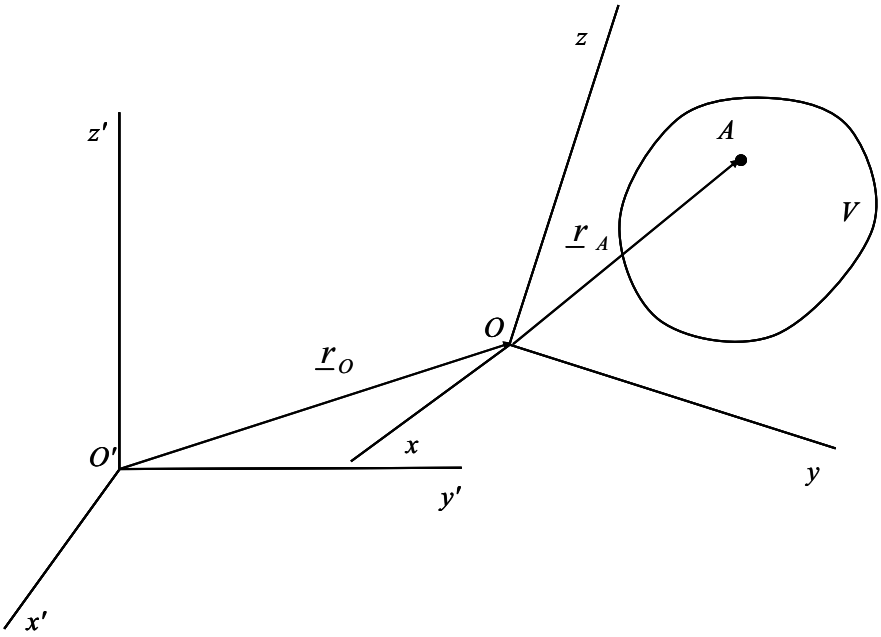

Fig. 1.12. Two frames for description of an arbitrary motion of a solid body

The kinetic torque of the solid body can be calculated as the integral over the volume of the body

$$\underline{K}_G = \int_V \rho\left(\underline{r}\right)\underline{r}\times\left(\underline{\Omega}\times\underline{r}\right)dV = \underline{\underline{\Theta}}\cdot\underline{\Omega} \tag{1.89}$$

Here $\rho\left(\underline{r}\right)$ is the density, $\underline{\underline{\Theta}}$ is the inertial tensor of the body. Then the relationship (1.81) takes the following form:

$$\underline{\underline{\Theta}}\cdot\Delta\underline{\Omega} = \sum_{i=1}^{n}\left(\underline{r}_i - \underline{r}_O\right)\times\underline{I}_i \tag{1.90}$$

Applying (1.88) and (1.90) to the equation (1.87) we can calculate the velocity of the arbitrary point A:

$$\Delta \underline{v}_A = \Delta \underline{v}_G + \Delta \underline{\Omega} \times \underline{r}_A = \frac{1}{M} \sum_{i=1}^{n} \underline{I}_i + \underline{\underline{\Theta}}^{-1} \cdot \sum_{i=1}^{n} \left(\underline{r}_i - \underline{r}_O \right) \times \underline{I}_i \qquad (1.91)$$

Here $\underline{\underline{\Theta}}^{-1}$ is the inverted tensor of inertia.

All these relationships enable us to calculate the jumps in the velocities if the contact forces or their impulses are known. Unfortunately it is not the case in collision problems. Thus some additional hypotheses are necessary.

1.4.4 Collinear Collision of Two Point Masses

Consider the simplest possible collision of two point masses moving along a certain straight line (Fig. 1.13).

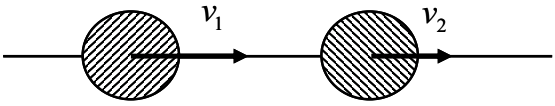

Fig. 1.13. Collision of two point masses

Assume we know their velocities just before the collision: v_1^- and v_2^-. Our objective is to determine the velocities of both masses after the collision. The equations of the impulsive motion for this simple case are:

$$\begin{aligned} m_1 \left(v_1(t) - v_1^- \right) &= -I(t) \\ m_2 \left(v_2(t) - v_2^- \right) &= I(t) \\ I(t) &= \int_{t_-}^{t} F(\tau) d\tau \end{aligned} \qquad (1.92)$$

The collision is possible if the masses approach each other before the collision (at the time t_-) and separate from each other after the collision (at the time t_+):

$$v_1^- > v_2^-;\ v_1^+ \le v_2^+ \qquad (1.93)$$

Between these time points there is an instant t_0 at which the relative velocity of the masses is equal to zero, i.e. $v_1(t_0) = v_2(t_0)$. This time separates two phases of the collision. Compression in the contact zone increases during the first phase. At the time t_0 it is maximal. After that the relaxation phase begins. It ends as soon as the contact force becomes equal to zero. The impulse at the end of the compression phase is

$$I(t_0) = \frac{m_1 m_2}{m_1 + m_2}\left(v_1^- - v_2^-\right) > 0 \tag{1.94}$$

During the collision the impulse can only increase, thus

$$I(t_+) \geq \frac{m_1 m_2}{m_1 + m_2}\left(v_1^- - v_2^-\right) \tag{1.95}$$

The fundamental hypothesis for the description of collisions was formulated by Newton, who has suggested that the relationship between the impulses depends on the material and shape of the colliding bodies but does not depend on their velocities before the collision.

He has introduced the impulse restitution coefficient R, which is still the basic characteristic of the collision.

$$I(t_+) = (1+R)\,I(t_0),\ 0 \leq R \leq 1 \tag{1.96}$$

Taking (1.94) into account we can calculate the final impulse:

$$I(t_+) = (1+R)\frac{m_1 m_2}{m_1 + m_2}\left(v_1^- - v_2^-\right) \tag{1.97}$$

The velocities after the end of the collision according to (1.92) are:

$$\begin{aligned} v_1^+ &= v_1^- - \frac{1}{m_1} I(t_+) = v_1^- - (1+R)\frac{m_2}{m_1 + m_2}\left(v_1^- - v_2^-\right) \\ v_2^+ &= v_2^- + \frac{1}{m_2} I(t_+) = v_2^- + (1+R)\frac{m_1}{m_1 + m_2}\left(v_1^- - v_2^-\right) \end{aligned} \tag{1.98}$$

Let us demonstrate that the coefficient R characterizes the energy losses during the collision. The change in the kinetic energy can be calculated according to the equation (1.86):

$$\begin{aligned} \Delta T &= -\frac{1}{2}\left(\left(v_1^+ + v_1^-\right) I(t_+) - \left(v_2^+ + v_2^-\right) I(t_+)\right) \\ &= -\frac{1}{2}\left(1 - R^2\right)\frac{m_1 m_2}{m_1 + m_2}\left(v_1^- - v_2^-\right)^2 \end{aligned} \tag{1.99}$$

The case $R = 1$ means that there are no energy losses during the collision. Such a collision is called absolutely elastic. If $R = 0$, the velocities of the collid-

ing bodies are equal after the collision. Such a collision is called absolutely inelastic or plastic.

Experiments show that the Newton's hypothesis correctly describes collisions in some intermediate range of the initial velocities (*cf.* [42]). The restitution coefficient is very close to one for glass balls ($R \approx 0.94$). It is almost equal to zero for heavy machines with a layer of bulk material between the colliding bodies.

1.4.5 Direct Collisions in Mechanical Systems with Ideal Constraints

The described approach can be easily generalized for a simple direct collision in a mechanical system with ideal, holonomic and steady constraints. A collision is called direct if the relative velocity of the colliding bodies is perpendicular to the contact surface [77]. Let us choose the coordinate q describing the distance between the contact surfaces as one of the generalized coordinates. The other coordinates are q_i, $i = 1, 2, \ldots, n$. The total kinetic energy of the system with $n+1$ degrees of freedom is a quadratic form of the generalized velocities:

$$
\begin{gathered}
T = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}\dot{\mathbf{q}} + \mathbf{m}^T \dot{\mathbf{q}}\dot{q} + \frac{1}{2} m\dot{q}^2 \\
\dot{\mathbf{q}} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}, \quad \mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & \ldots & m_{1n} \\ m_{21} & m_{22} & \ldots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \ldots & m_{nn} \end{bmatrix}
\end{gathered}
\tag{1.100}
$$

Equations of motion for this system are:

$$
\begin{gathered}
\dot{\mathbf{p}} - \frac{\partial T}{\partial \mathbf{q}} = \mathbf{Q}, \qquad \mathbf{p} = \frac{\partial T}{\partial \dot{\mathbf{q}}} = \mathbf{M}\dot{\mathbf{q}} + \mathbf{m}\dot{q} \\
\dot{p} - \frac{\partial T}{\partial q} = Q + F, \qquad p = \frac{\partial T}{\partial \dot{q}} = \mathbf{m}^T \dot{\mathbf{q}} + m\dot{q} \\
\mathbf{p} = \begin{bmatrix} p_1 & p_2 & \ldots & p_n \end{bmatrix}^T, \\
\frac{\partial T}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial T}{\partial q_1} & \frac{\partial T}{\partial q_2} & \ldots & \frac{\partial T}{\partial q_n} \end{bmatrix}^T, \quad \mathbf{Q} = \begin{bmatrix} Q_1 & Q_2 & \ldots & Q_n \end{bmatrix}^T
\end{gathered}
\tag{1.101}
$$

Here Q is the generalized force corresponding to the contact coordinate, F is the corresponding contact force, $\mathbf{Q}$ is the vector of the generalized forces corre-

sponding to other coordinates. The generalized forces do not contain reactions of the constraints and thus are finite during collision.

Integrating these equations during the time interval of the collision we obtain:

$$\mathbf{p}(t)-\mathbf{p}_{-}=0;\; p(t)-p_{-}=I(t) \tag{1.102}$$

Taking (1.101) into account we obtain for the generalized velocities the following linear equations:

$$\begin{aligned}&\mathbf{M}\left(\dot{\mathbf{q}}(t)-\dot{\mathbf{q}}_{-}\right)+\mathbf{m}\left(\dot{q}(t)-\dot{q}_{-}\right)=0\\&\mathbf{m}^{T}\left(\dot{\mathbf{q}}(t)-\dot{\mathbf{q}}_{-}\right)+m\left(\dot{q}(t)-\dot{q}_{-}\right)=I(t)\end{aligned} \tag{1.103}$$

These equations can be easily solved with respect to the change of the generalized velocities:

$$\left.\begin{aligned}&\dot{\mathbf{q}}(t)-\dot{\mathbf{q}}_{-}=-\mathbf{M}^{-1}\mathbf{m}\left(\dot{q}(t)-\dot{q}_{-}\right)\\&\left(m-\mathbf{m}^{T}\mathbf{M}^{-1}\mathbf{m}\right)\left(\dot{q}(t)-\dot{q}_{-}\right)=I(t)\end{aligned}\right\}\Rightarrow$$

$$\dot{q}(t)=\dot{q}_{-}+\frac{I(t)}{m-\mathbf{m}^{T}\mathbf{M}^{-1}\mathbf{m}} \tag{1.104}$$

$$\dot{\mathbf{q}}(t)=\dot{\mathbf{q}}_{-}-\mathbf{M}^{-1}\mathbf{m}\frac{I(t)}{m-\mathbf{m}^{T}\mathbf{M}^{-1}\mathbf{m}}$$

These relationships are valid for the whole time interval of the collision. At the end of the compression phase in particular, we obtain the following relationship:

$$\dot{q}(t_0)=0\Rightarrow I(t_0)=-\left(m-\mathbf{m}^{T}\mathbf{M}^{-1}\mathbf{m}\right)\dot{q}_{-} \tag{1.105}$$

Applying the Newton's hypothesis (1.96) we can calculate the final impulse:

$$I(t_{+})=(1+R)I(t_0)=-(1+R)\left(m-\mathbf{m}^{T}\mathbf{M}^{-1}\mathbf{m}\right)\dot{q}_{-} \tag{1.106}$$

Finally we find all the generalized velocities after the collision:

$$\begin{aligned}&\dot{q}_{+}=-R\dot{q}_{-}\\&\dot{\mathbf{q}}_{+}=\dot{\mathbf{q}}_{-}+(1+R)\mathbf{M}^{-1}\mathbf{m}\dot{q}_{-}\end{aligned} \tag{1.107}$$

This result completes the analysis of the collision of the mechanical system with ideal constraints.

1.4.6 Concluding Remarks

Only the simplest cases of the direct collisions were considered in this paragraph. Analysis in case of a general collision, i.e. if the relative velocity has a tangential component to the contact surface, is much more complex. Especially important is to take the friction forces in the contact zone into account, because the friction forces are proportional to the normal contact force, i.e. they also have the impulsive character. A compact analysis of these problems from the axiomatic point of view can be found in [77].

However a different approach is useful in applications and especially in numerical simulations. It is based on the nonlinear contact elements taking energy dissipation and friction into account. These elements can be used almost universally in order to describe complex collisions with multiple contacts. However it is not easy to determine their parameters [52].

2. Oscillations in Systems with Dry Friction

Dry friction is one of the most complex phenomena in mechanics. Generalizing one can say that "dry" friction is the macroscopic appearance of the microscopic processes in a thin layer including the sliding surfaces and liquid or gas between them. The detailed modeling of these processes depends on the micro-conditions and may be extremely sophisticated. Some of the corresponding investigations and additional references can be found in [2, 43]. For example in many cases it seems to be necessary to take wear into account in order to explain some important features of friction.

Two basic effects of the dry friction are taken into account in all theories. The first of them is the discontinuity of dry friction which is connected with the difference between resistance to sliding whilst stationary (*stick, static friction*), and resistance to sliding in motion (*slip, dynamic friction*). The ability of a friction contact to resist against an applied external force without any macro motion displays the fact that friction can be interpreted as a constraint in such a situation.

The contact resistance against slip is usually characterized through the friction coefficient. The friction coefficient is the relation between the magnitude of the friction force and the magnitude of the normal pressure force in contact. This coefficient is however not constant. It depends at least on the relative velocity between the contacting surfaces and usually on the normal force (or normal pressure) itself.

Many mechanical interfaces are characterized by a form of dry friction where the force-velocity curve has negative slope at low velocities. Initially, friction decreases as the contacting objects start to move, whereas at higher velocities the friction force increases again; in particular this characterizes surfaces with boundary lubrication. The initial negative slope corresponds to negative damping and may thus cause oscillations that grow in amplitude, until a balance of dissipated and induced energy is attained, as pointed out already by Lord Rayleigh [108].

The characteristic change in friction coefficient with velocity has been explained quite convincingly by Tolstoi [127], who considers the normal separation distance between the friction surfaces as a key to the specific shape of the friction curve. However it is currently unclear to which extent the friction-velocity relationship, obtained experimentally during a quasi-static change of velocity, can be used to describe friction forces in dynamics, for example during the stationary oscillations [132].

A typical measured friction curve for low relative velocities is shown in Fig. 2.1. It was measured for two steel bodies in oil. One can easily see the typical negative gradient of the friction coefficient at low relative velocities. The viscose properties of the lubricant become dominant at higher relative velocities and result

in increasing friction coefficient. (This situation is not shown in Fig. 2.1 but can be found in literature [2].)

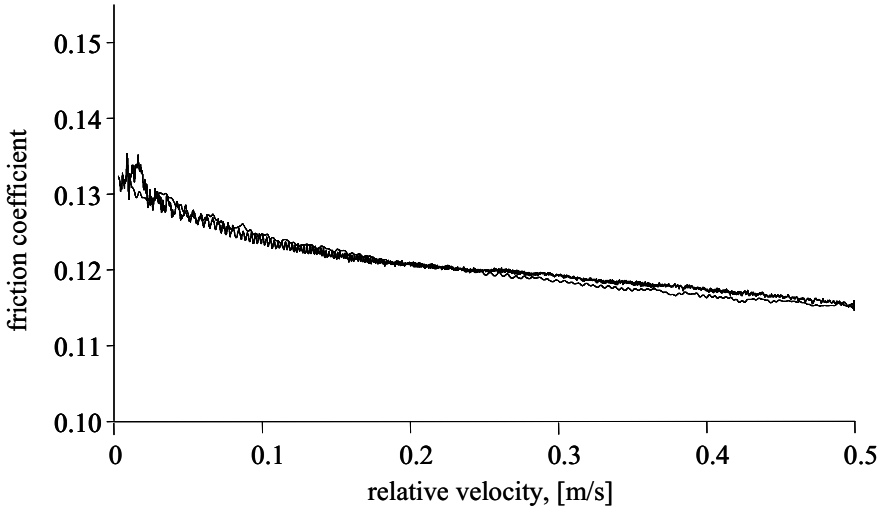

Fig. 2.1. A typical measurement of the friction coefficient

There are three main phenomena connected with the dry friction in oscillating systems: *stick-slip oscillations* due to negative friction gradient, instabilities due to the *non-conservative character of friction* and the *friction induced displacement*.

The stick-slip vibrations are well known in many kinds of engineering systems and everyday life, e.g. as sounds form when a violin is played, squeaking chalks and shoes, creaking doors, squealing tramways, chattering machine tools, and grating brakes. Numerous works are devoted to the study of friction induced oscillations. For ease of setup and interpretation an idealized physical system consisting of a mass sliding on a moving belt has been considered very often. Panovko and Gubanova [85] show that self-excited oscillations occur in such a system only when the belt velocity is lower than the value corresponding to the minimum of the friction coefficient. Tondl [128], Nayfeh and Mook [80] and Mitropolskii and Nguyen [75] describe self-excited oscillations of the "*mass-on-moving-belt*" system, presenting approximate expressions for the vibration amplitudes for the case where there is no sticking between mass and belt. Popp [102] presents models with numerical and experimental results for four systems that are similar to the mass-on-moving-belt. Ibrahim [49, 50] and McMillan [70] present and discuss the basic mechanics of friction and friction models and provide reviews on relevant literature. A very readable historical review on dry friction and stick-slip phenomena is given by Guran *et al.* [45] and a large survey on friction literature until 1992 by Armstrong-Hélouvry *et al.* [5].

Much research seems to be concentrated on determining the onset of stick-slip vibrations in order to avoid these totally. However, stick-slip vibrations might be acceptable in applications, provided their amplitudes are sufficiently small. Therefore, simple expressions providing an immediate insight into the influence of pa-

rameters on vibration amplitudes are useful. There are, however, very few works providing expressions for stationary stick-slip amplitudes. Armstrong-Hélouvry [6] performed a perturbation analysis for a system with *Stribeck* friction and frictional lag, predicting the onset of stick-slip for a robot arm. Elmer [30] discusses stick-slip and *pure slip oscillations* of the mass-on-moving-belt system with no damping and different kinds of friction functions, provides analytical expressions for the transfer between stick-slip and pure-slip oscillations, and sketches typical local and global bifurcation scenarios. Thomsen [122] sets up approximate expressions for stick-slip oscillations of the mass-on-moving-belt, which are accurate for very small differences in static and dynamic friction. In section 2.1 we generalize these results predicting the stationary amplitudes for larger differences in static and dynamic friction (*cf.* [125]). We also use this classical example in order to introduce two different approaches for the analysis of discontinuous systems. The basic idea of the first one, which can be called "*stitching*", is to split the considered motion into several time intervals. In each of these intervals the system's behavior is continuous and all the discontinuities are replaced by the switching conditions between different time intervals and subsystems. The second one is averaging; its applications in this chapter are limited to the pure slip motions.

The standard publications are concentrated so strongly on the negative friction gradient as the source for friction induced instability that other mechanisms are seemingly almost forgotten. These mechanisms are possible in systems with two ore more degrees of freedom which become coupled due to the fact that dry friction substantially couples the normal and tangential contact forces. Several examples for these types of the friction induced instabilities can be found in [28, 105]. The applications include gear rattle, brake squeal and others. A simple example illustration of this type of instability is discussed in section 2.2.

The third phenomenon connected with dry friction is the vibration induced displacement and transportation. Intensive research in this area was started in a 1964 the book by Blekhmann and Dzhanelidze [15]. Many industrial applications for this phenomenon, like vibrating conveyers and screens followed this book (see for example [18]). A particular problem of the so-called "Chelomei's pendulum" [27], combined two interesting effects – a pendulum under a high frequency excitation, and a vibration induced displacement. It attracted for a long period of time attention of different scientists [16, 20, 124].

The recent interest to such systems was inspired by the development of electronic devices combined with piezo-electric ceramics and ultrasonic motors as excitation sources for the mechanical part of a system [130]. Several works from the Danish school should be mentioned here [53, 73, 118, 123, 124], where the Direct Separation of Motions (*cf.* [20]) was used for systems with friction. But in particular cases, where the first approximation is not sufficient, special methods were developed and successfully used by Fidlin and Thomsen [37, 40]. These examples are discussed in sections 2.3 and 2.4.

2.1 Self Excited Oscillations of the Mass-on-Moving-Belt

2.1.1 The Problem Description; Equations of Motion

Fig. 2.2 shows the physical system we are going to investigate in this section. It is a mass M on a belt which moves at constant speed V_b. The mass is a rigid body; its position at time $\tilde{t}$ is $X(\tilde{t})$ in a fixed frame of coordinates. It is subjected to a normal static pressure load F, linear spring-loading CX, damping force $b\,dX/d\tilde{t}$ and a friction force $F\mu(V_r)$. The dynamic friction coefficient μ here is a function of the relative velocity, $V_r = dX/d\tilde{t} - V_b$; the static friction force is $F\mu_s$.

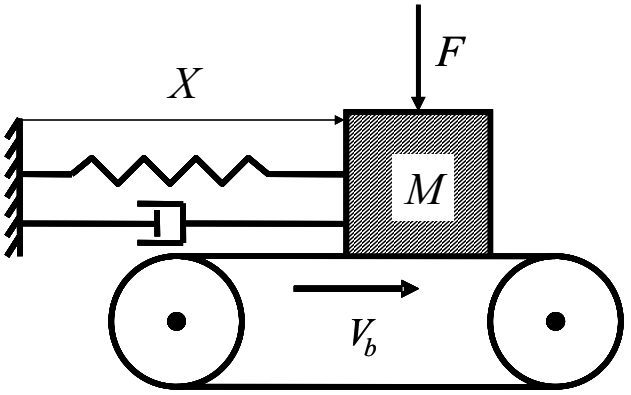

Fig. 2.2. The classical "mass-on-moving-belt" system

The motion of the system is governed by the following equations (in non-dimensional form):

$$\ddot{x} + 2\beta\dot{x} + x + \mu(\dot{x} - v_b) = 0 \text{ if } \dot{x} \neq v_b \quad (\text{slip}) \tag{2.1}$$

$$\ddot{x} = 0,\ x + 2\beta v_b < \mu_s \text{ if } x = v_b \quad (\text{stick}) \tag{2.2}$$

Here already we have to distinguish between two qualitatively different states of the system. When the mass slips relative to the moving belt its motion is governed by the equation (2.1). But if the conditions (2.2) are fulfilled, the mass can stick to the belt due to the static friction force.

All parameters in the equations are positive and both variables and parameters are undimensioned according to the following relationships:

$$L=\frac{F}{C};\ x=\frac{X}{L};\ \omega_0=\sqrt{\frac{C}{M}};\ t=\omega_0\tilde{t};\ v_b=\frac{V_b}{\omega_0 L};\ \beta=\frac{b}{2\sqrt{CM}}$$
$$\dot{x}=\frac{dx}{dt} \tag{2.3}$$

Here lengths have been normalized by the characteristic length L and time by the linear natural frequency ω_0 of free oscillations of the mass when there is no damping and friction.

For the friction function $\mu(v_r)$ we assume, following to [49, 50, 85] (*cf.* Fig. 2.3):

$$\mu(v_r)=\mu_s\,\mathrm{sgn}(v_r)-k_1 v_r+k_3 {v_r}^3 \tag{2.4}$$

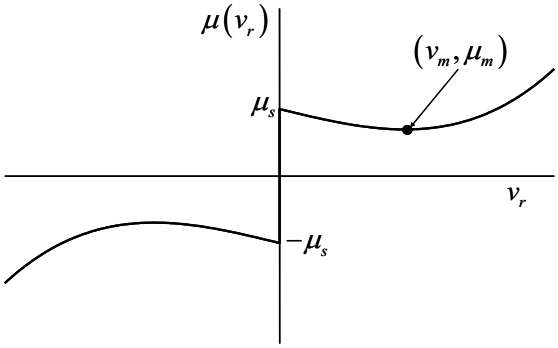

Fig. 2.3. Friction coefficient as a function of the relative velocity

The following signs are used in the equation (2.4):

$$v_r=\dot{x}-v_b$$
$$k_1=\frac{3}{2}\frac{\mu_s-\mu_m}{v_m};\ k_3=\frac{1}{2}\frac{\mu_s-\mu_m}{{v_m}^3} \tag{2.5}$$

Here μ_s is the coefficient of static friction, v_m is the velocity corresponding to the minimum coefficient μ_m of dynamic friction, $\mu_m<\mu_s$, and $k_1>0,\ k_3>0$. As it appears $|\mu|<\mu_s$ when the mass is at rest on the moving belt ($v_r=0$, *stick phase*), whereas when the mass starts sliding the friction forces initially decrease with increasing velocity ($v_r\neq 0$, *slip phase*). This particular form of the friction law is not overly restricted; it resembles characteristic features of friction models in common use. The so-called "Stribeck friction", describing the friction-velocity

relationship for systems with boundary lubrication, also shares the essential features of (2.4), even though they differ in detail.

The problem to be solved below is to determine stable periodic solutions to (2.1), (2.2) with friction law (2.4). Closed form solutions are not available, due to the discontinuity and nonlinearity of the friction function μ. However, for the important case of relatively small difference between static and dynamic friction coefficients, we can employ perturbation analysis in order to set up approximate analytical expressions, and check the validity of results by using numerical simulation.

2.1.2 Types of Motion

According to (2.1) and (2.4) the mass has a static equilibrium at $x = \overline{x}$

$$\overline{x} = -\mu(-v_b) = \mu_s - k_1 v_b + k_3 v_b^{\,3} \tag{2.6}$$

To study motions near this equilibrium we shift the origin by defining the new variable.

$$u(t) = x(t) - \overline{x} \tag{2.7}$$

Equations (2.1) and (2.2) are transformed by this substitution into

$$\ddot{u} + u + \varepsilon h(\dot{u}) = 0 \qquad \text{(slip)} \tag{2.8}$$

$$\ddot{u} = 0, \quad u - \left(k_1 v_b - k_3 v_b^{\,3}\right) + 2\beta v_b < 0 \quad \text{for} \quad \dot{u} = v_b \quad \text{(stick)} \tag{2.9}$$

The following function is introduced here:

$$\begin{aligned} \varepsilon h(\dot{u}) &\equiv 2\beta\dot{u} + \mu(\dot{u} - v_b) - \mu(-v_b) \\ &= \mu_s \left(1 + \operatorname{sgn}(\dot{u} - v_b)\right) + (2\beta - k_1 + 3k_3 v_b^{\,2})\dot{u} \\ &\qquad -3k_3 v_b \dot{u}^2 + k_3 \dot{u}^3 \end{aligned} \tag{2.10}$$

Here $\varepsilon << 1$ has been introduced as a formal book-keeping parameter, in order to indicate that the damping and the difference in static and kinetic friction coefficient are assumed small. The real assumptions are: the parameters k_3 and $2\beta - k_1$ are small, the non-dimensional belt's velocity has the magnitude order 1. For example we can choose this formal parameter as follows: $\varepsilon = \max\left\{\left|k_1 - 2\beta\right|, \left|k_3\right|\right\}$.

The equilibrium $u = 0;\ \dot{u} = 0$ in (2.8) corresponds to a steady state sliding, with the mass being at rest and the belt sliding at constant velocity v_b below it.

Expression (2.10) shows explicitly, that the negative friction gradient produces negative damping. It is the simplest source for instability in dynamic systems. Hence, this static equilibrium can be stable or unstable. If it is unstable, then stable periodic motion takes over; this is the only possibility, since generally the steady state must be a static equilibrium, a periodic motion, or a chaotic motion – and chaotic solutions cannot occur for a single second order autonomous ordinary differential equation (e.g. [120]).

Two different kinds of periodic solutions to (2.8), (2.9) are considered below: pure slip oscillations where the velocity of the mass is always smaller than the velocity of the belt – and stick-slip oscillations where the velocities of the mass can be equal to (i.e. the mass occasionally sticks to the belt) or even exceed the velocity of the belt. The pure-slip oscillations are shown in Fig. 2.4 (velocity as the time series and the corresponding phase diagram).

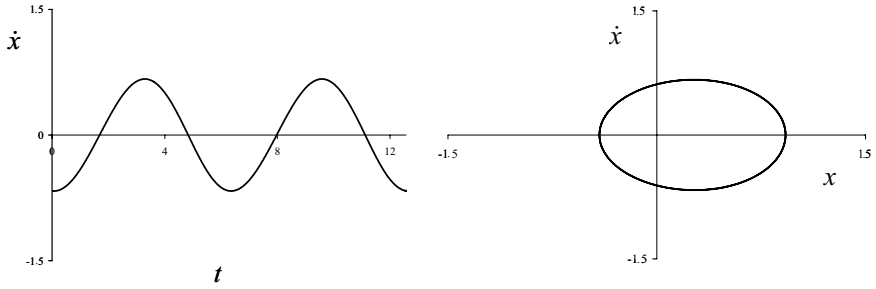

Fig. 2.4. Pure slip oscillations of the mass-on-moving-belt

The sick-slip oscillations are shown in Fig. 2.5.

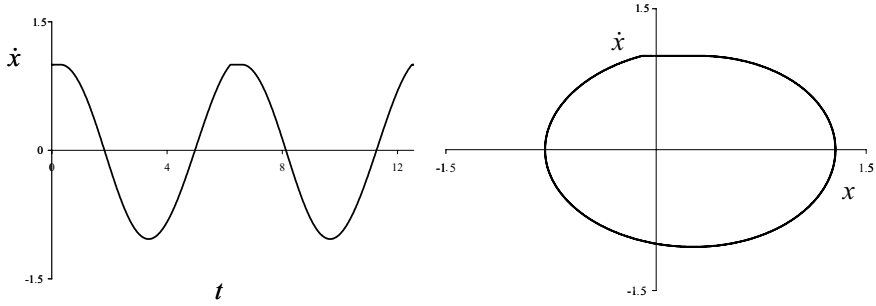

Fig. 2.5. Stick-slip oscillations of the mass-on-moving-belt

It seems that during stationary oscillations the velocity of the mass would never exceed that of the belt, i.e. $\dot{u}(t) \le v_b$ for $t > t_0$. This is so because the energy storing spring cannot accelerate the mass to a velocity exceeding the maximum velocity during the previous oscillation period, and the energy-providing belt cannot accelerate the mass to a velocity beyond its own. It is of course, possible to

start the system from a state with $\dot{u} > v_b$, however, viscous damping and dry friction will then drain energy until a stationary state is achieved with $\dot{u} \le v_b$. This explanation is however not quite correct. The negative friction gradient means qualitatively nothing other than the negative damping which is active during the motion of the mass relative to the belt. Thus the energy of the mass can increase during slipping. So it can overtake the belt for a short period of time. But than, the direction of the relative velocity changes and the friction decelerates the mass until it sticks. The corresponding motion is illustrated in Fig. 2.6.

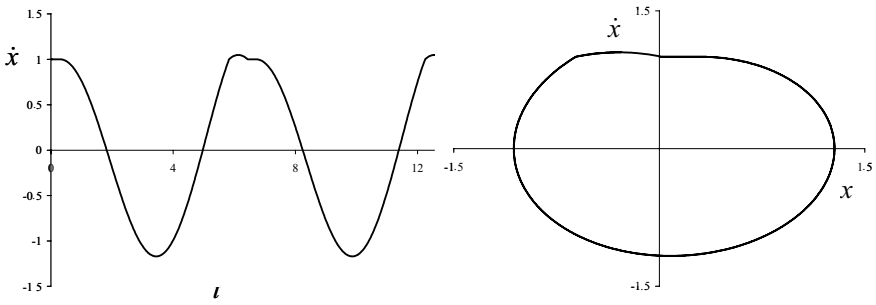

Fig. 2.6. Stick-slip oscillations of the mass overtaking the belt

2.1.3 Pure Slip Oscillations

With pure slip the inequality $\dot{u} < v_b$ is valid for all t, so that the discontinuity of the friction function is never met or crossed. For this case the function εh in (2.10) can be written as follows:

$$\begin{aligned} &\varepsilon h(\dot{u}) = h_1\dot{u} + h_2\dot{u}^2 + h_3\dot{u}^3 \qquad \text{for} \quad \dot{u} < v_b \\ &h_1 = 2\beta - k_1 + 3k_3 {v_b}^2; \quad h_2 = -3k_3 v_b; \quad h_3 = k_3 \end{aligned} \tag{2.11}$$

In order to apply the standard averaging for solving (2.8) the standard Van-der-Pol transformation based on the harmonic solution to the unperturbed problem can be used:

$$u = A\sin\psi; \qquad \dot{u} = A\cos\psi \tag{2.12}$$

The new variables $A(t)$ and $\psi(t)$ are governed by a system in standard form for averaging:

$$\begin{aligned}\dot{A} &= -\varepsilon h\left(A\cos\psi\right)\cos\psi \\ \dot{\psi} &= 1+\frac{\varepsilon}{A}h\left(A\cos\psi\right)\sin\psi\end{aligned} \tag{2.13}$$

Averaging this system with respect to the fast phase ψ we obtain an equation of the first order approximation:

$$\dot{A} = -\frac{1}{2}\varepsilon A\left(h_1+\frac{3}{4}h_3A^2\right) \tag{2.14}$$

There are two equilibrium points in this equation: A trivial solution $A=0$ corresponding to the static equilibrium $u=0$ and a nontrivial solution given by

$$A_1 = \sqrt{-\frac{4h_1}{3h_3}} \tag{2.15}$$

This stationary amplitude corresponds to periodic solutions $u = A_1\sin\left(t+\theta\right)$ with an arbitrary phase θ (the considered system is autonomous).

As for the stability of solutions, one finds that the trivial solution is unstable if the friction gradient is negative; taking (2.11) and (2.5) into account this condition becomes:

$$v_b < v_{b1} \equiv v_m\sqrt{1-\frac{4\beta v_m}{3\left(\mu_s-\mu_m\right)}} \tag{2.16}$$

As it appears, when there is no viscous damping ($\beta=0$) the static equilibrium is unstable for excitation speeds lower than v_m. Viscous damping stabilizes the equilibrium, and at sufficiently large damping $\beta > \frac{3}{4}(\mu_s-\mu_m)$, the static equilibrium is always stable.

Periodic motions exist and are stable if $h_1<0$ and $h_3>0$. Since $\mu_s>\mu_m$, the latter requirement is automatically fulfilled. The amplitude A_1 of the stable periodic motion can be found by inserting (2.11) and (2.5) into (2.15):

$$A_1 = 2v_m\sqrt{1-\left(v_b/v_m\right)^2-\frac{4\beta v_m}{3\left(\mu_s-\mu_m\right)}} = 2\sqrt{{v_{b1}}^2-{v_b}^2} \tag{2.17}$$

$$v_{b0} < v_b < v_{b1}$$

Here v_{b1} is the speed below which pure slip oscillations first occur, as given by (2.16).

This expression for A_1 assumes pure slip, so the increase in amplitude for decreasing v_b will cease when the mass starts sticking to the belt, i.e. when $\max(\dot{u}) = v_b$. With $\max(\dot{u}) = A_1$ (by (2.12)), it is found that sticking first occurs when $A_1 = v_b$. Inserting this into (2.17) and solving for v_b we find that stick-slip oscillations occur when $v_b < v_{b0}$, where

$$v_{b0} = \sqrt{\frac{4}{5}} v_{b1} \tag{2.18}$$

Hence, the range of belt velocities where pure slip oscillations occur is rather small, its width $(v_{b1} - v_{b0})$ being only $1 - \sqrt{4/5} \approx 10\%$ of v_{b1}. It forms a transition zone to a wider range of belt velocities where stick-slip motions occur.

When sticking just starts, the amplitude of oscillations is given by inserting $v_b = v_{b0}$ in (2.17) and then using (2.16) and (2.18) to find:

$$A_1\big|_{v_b = v_{b0}} = A_{1,\max} = v_{b0} = \sqrt{\frac{4}{5}} v_m \sqrt{1 - \frac{4\beta v_m}{3(\mu_s - \mu_m)}} \tag{2.19}$$

Hence, for vanishing damping β the maximum amplitude grows linearly with the velocity v_m of minimum kinetic friction. As it appears from (2.12), the non-dimensional displacement amplitude equals the velocity amplitude; thus the velocity amplitude $A_{v1} = A_1$.

2.1.4 Stick-slip Oscillations

Let us investigate what happens if the velocity of the belt is smaller that the critical value calculated according to (2.18), i.e. $v_b < v_{b0}$. We have shown that in that case the mass sticks to the belt during a part of an oscillation period.

This case cannot be analyzed using the above averaging procedure for pure slip oscillations, since the switch from slip to stick is accompanied by the change of number of degrees of freedom in this system from one to zero. As long as the mass sticks to the belt its motion is determined kinematically and it does not have any degree of freedom. While slipping it has one degree of freedom and its motion is determined by the corresponding equation of dynamics.

The simplest way to solve this problem is to analyze the stick and the slip phases of the motion separately, and link the results together to obtain an approximate expression for one full oscillation period. Let us start with the slip phase.

Slip phase

During sticking the mass moves together with the belt, i.e. $\dot{u} = v_b$. This continues until the force from the restoring spring and the damper has increased to the maximum static friction force, i.e. until the strict inequality in (2.2) or (2.9) is no longer satisfied. We consider this the initial condition at time $t = 0$, where the stick phase ends and the slip phase begins, i.e.:

$$u(0) = -2\beta v_b + k_1 v_b - k_3 v_b^3; \quad \dot{u}(0) = v_b; \qquad \text{(slip starts)} \tag{2.20}$$

Motions during the subsequent slip phase are then governed by (2.8). This equation is non-linear, so approximate methods are in need. However, during slip it is continuous in its highest derivatives, since the inequality $\dot{u} < v_b$ holds during the slip phase. Further, since the solution is only needed for the finite time interval of the slip phase we can use a straightforward perturbation approach. We can try to find an approximate solution as follows:

$$u(t) = u_0(t) + \varepsilon u_1(t), \qquad t \in [0; t_{s1}], \qquad \varepsilon << 1 \tag{2.21}$$

We substitute this relationship into (2.8) and (2.20), balance terms of the same powers of ε , insert (2.10), and obtain two new initial value problems determining the functions u_0 and u_1 :

$$\ddot{u}_0 + u_0 = 0; \qquad u_0(0) = 0; \quad \dot{u}_0(0) = v_b \tag{2.22}$$

$$\begin{aligned} &\ddot{u}_1 + u_1 = -h(\dot{u}_0); \quad u_1(0) = -2\beta_* v_b + k_{1*} v_b - k_{3*} v_b^3; \quad \dot{u}_1(0) = 0 \\ &\beta_* = \frac{2\beta}{\varepsilon}; \; k_{1*} = \frac{k_1}{\varepsilon}; \; k_{3*} = \frac{k_3}{\varepsilon} \end{aligned} \tag{2.23}$$

The solution to (2.22) for the zero-order approximation u_0 is:

$$u_0 = v_b \sin(t) \tag{2.24}$$

Inserting this and (2.10) into (2.23), the equation for the first order correction u_1 can be obtained:

$$\ddot{u}_1 + u_1 = \frac{3}{2}c_3 - c_1\cos(t) + \frac{3}{2}c_3\cos(2t) - \frac{1}{4}c_3\cos(3t) \tag{2.25}$$

Here we have used the following signs:

$$c_3 \equiv k_{3*}v_b^{\,3};\; c_1 \equiv 2\beta_* v_b - k_{1*}v_b + \frac{15}{4}c_3 \tag{2.26}$$

The solution of this linear equation satisfying the initial conditions in (2.23) is:

$$u_1 = \frac{3}{2}c_3 - \frac{1}{2}c_1 t\sin(t) + \left(\frac{55}{32}c_3 - c_1\right)\cos(t) - \frac{1}{2}c_3\cos(2t) + \frac{1}{32}c_3\cos(3t) \tag{2.27}$$

The secular term $t\sin t$ is fully acceptable here, since it remains bounded in the finite time of slipping.

Hence, the motion during the slip phase is approximately given by:

$$u(t) = v_b\sin t + \varepsilon\left\{\frac{3}{2}c_3 - \frac{1}{2}c_1 t\sin t + \left(\frac{55}{32}c_3 - c_1\right)\cos t - \frac{1}{2}c_3\cos 2t + \frac{1}{32}c_3\cos 3t\right\} + O\left(\varepsilon^2\right) \tag{2.28}$$

$t \in [0, t_{s1}]$ (slip phase)

Expression $O\left(\varepsilon^2\right)$ denotes here small terms. The corresponding velocities $\dot{u}$ and accelerations $\ddot{u}$ can be obtained simply by differentiation.

It is still necessary to determine t_{s1}, the time where slip stops and stick can start. This occurs after the mass has slipped back on the belt, and has been accelerated forward by the spring and the friction, until the velocity of the mass again equals the belt's speed. Hence we determine t_{s1} as the first solution of the equation $\dot{u}\left(t_{s1}\right) = v_b$ for $t_{s1} > 0$. Using (2.28) directly to compute $\dot{u}$, a transcendental equation would have to be solved numerically to determine t_{s1}. The most consistent way to estimate its solution seems to be the asymptotic analysis used above. So the zero order approximation according to (2.24) is $v_b\cos t_s = v_b$, with solution:

$$t_{s1}^0 = 2\pi \tag{2.29}$$

The perturbed equation (2.28) should then determine the first order approximation for t_{s1}. However, (2.29) is an extreme point of the cosine function, and so the accuracy of the first approximation will be $O(\sqrt{\varepsilon})$. Hence, for the typical values of friction differences used in this study, i.e. $\mu_s - \mu_m \approx 0.3$, the error in t_{s1} will be about 55%. This problem especially concerns t_{s1}, whereas vibration amplitudes are fairly well approximated (see below). If a better approximation for t_{s1} is needed, then one could consider the higher-level approximations for the solutions of (2.8). It should be recalled that numerical solutions for t_{s1} can always be obtained simply by solving the algebraic equation $\dot{u}(t_{s1}) = v_b$ numerically, with $\dot{u}$ given by differentiation of (2.28):

$$\dot{u}(t) = v_b \cos t + \varepsilon \left\{ -\frac{1}{2} c_1 (\sin t + t \cos t) - \left(\frac{55}{32} c_3 - c_1 \right) \sin t + c_3 \sin 2t - \frac{3}{32} c_3 \sin 3t \right\} + O(\varepsilon^2) \tag{2.30}$$

$$t \in [0, t_{s1}] \qquad \text{(slip phase)}$$

The first order approximation to t_{s1} can be found as follows:

$$t_{s1}^1 = 2\pi - \sqrt{\varepsilon}\tau_{s1}^1 + O(\varepsilon^{3/2})$$
$$\tau_{s1}^1 = \sqrt{-\frac{2c_1}{v_b}} = \sqrt{2k_{1*} - 4\beta_* - \frac{15}{2} k_{3*} {v_b}^2} > 0 \tag{2.31}$$

Substituting this relationship into the sticking condition (2.9) we can transform this inequality to the following form:

$$-\sqrt{\varepsilon}\tau_{s1} + \varepsilon \left(2\beta_* - k_{1*} + k_{3*} {v_b}^2 \right) \le 0 \tag{2.32}$$

If our formal small parameter ε is sufficiently small, this condition is always fulfilled because $\sqrt{\varepsilon} >> \varepsilon$. The corresponding motion was illustrated in Fig. 2.5.

If, however, ε is sufficiently large, then this inequality is not fulfilled. This situation was illustrated in Fig. 2.6. In this book we remain in the frame of perturbation analysis and assume the small parameter to be sufficiently small.

Stick Phase

Stick starts at $t = t_{s1}$ and then the mass just follows the belt, i.e.

$$\dot{u} = v_b;\ u(t) = u(t_s) + v_b(t - t_s);\ \ t \in (t_s; T) \tag{2.33}$$

Here $u(t_{s1})$ is known by having applied (2.28) for the just completed phase of slip. The time $t = T$ where stick ends is determined by the periodicity condition, i.e. $u(T) = u(0)$, so that

$$T = t_{s_1} + \frac{u(0) - u(t_{s1})}{v_b} \tag{2.34}$$

Stick-Slip Vibration Amplitude

Considering velocity amplitudes during one stick-slip cycle, we note that during stick the velocity is constant, $\dot{u} = v_b$ while during slip the velocity changes continuously with a maximum absolute value at time $t = t_m$ defined by the solution to $\ddot{u}(t_m) = 0$. Seeking an approximate solution, we let

$$t_m = t_{m0} + \varepsilon t_{m1};\quad \varepsilon << 1 \tag{2.35}$$

Inserting this into (2.28) and Taylor-expanding for small ε, one finds:

$$\ddot{u}(t_m) = \ddot{u}_0(t_{m0}) + \varepsilon\left(t_{m1}\dddot{u}_0(t_{m0}) + \ddot{u}_1(t_{m0})\right) + O(\varepsilon^2) \tag{2.36}$$

Balancing terms of same orders of magnitude, it can be found that the condition $\ddot{u}(t_m) = 0$ is approximately satisfied if

$$\ddot{u}_0(t_{m0}) = 0;\quad \ddot{u}_1(t_{m0}) = -t_{m1}\dddot{u}_0(t_{m0}) \tag{2.37}$$

Considering (2.24) it can be noticed that the first equation is satisfied by $t_{m0} = \pi$. Inserting this and relationships (2.24), (2.27) for u_0, u_1 into the second equation, one can determine the first order approximation (letting $\varepsilon = 1$):

$$t_m = \pi - \frac{c_1}{v_b} - \frac{73}{32}\frac{c_3}{v_b} \tag{2.38}$$

The velocity at this time is:

$$\dot{u}(t_m) = -v_b + \frac{\pi}{2} c_1 \tag{2.39}$$

As a measure indicating the magnitude of oscillations, which are asymmetric with respect to $u = 0$, we can use the velocity amplitude A_{v0} of stick-slip oscillations as half the peak-to-peak velocity, i.e.

$$A_{v0} = \frac{1}{2}\left(v_b - \dot{u}(t_m)\right) \tag{2.40}$$

This velocity can be calculated as follows:

$$A_{v0} = \left(1 - \frac{\pi}{2}\beta\right) v_b + \frac{3\pi}{8}\left(\mu_s - \mu_m\right)\frac{v_b}{v_m}\left(1 - \frac{5}{4}\left(\frac{v_b}{v_m}\right)^2\right), \quad v_b < v_{b0} \tag{2.41}$$

The corresponding displacement amplitude A_0 can be determined similarly, by calculating approximations to the times t_{m+} and t_{m-} where $\dot{u} = 0$; (*cf.* Fig.2.7, which shows displacement, velocity, and acceleration during one cycle of stick-slip oscillation).

The corresponding approximation is

$$\begin{aligned} t_{m-} &= \frac{3\pi}{2} - \frac{1}{2}c_1 + \frac{52}{32}c_3; \qquad u(t_{m-}) = -v_b + 2c_3 + \frac{3\pi}{4}c_1 \\ t_{m+} &= \frac{\pi}{2} + \frac{1}{2}c_1 - \frac{52}{32}c_3; \qquad u(t_{m+}) = v_b + 2c_3 - \frac{\pi}{4}c_1 \end{aligned} \tag{2.42}$$

Defining the displacement amplitude A_1 as half the peak-to-peak displacement, we become the following result:

$$\begin{aligned} A_0 &= \frac{1}{2}\left(u(t_{m+}) - u(t_{m-})\right) = v_b - \tfrac{\pi}{2}c_1 \\ &= \left(1 - \pi\beta\right) v_b + \frac{3\pi}{4}\left(\mu_s - \mu_m\right)\frac{v_b}{v_m}\left(1 - \frac{5}{4}\left(\frac{v_b}{v_m}\right)^2\right), \qquad v_b < v_{b0} \end{aligned} \tag{2.43}$$

One can show that this function has a maximum value at $v_b = v_b^*$:

$$v_b^* = \begin{cases} v_{b0}, & \text{for } \dfrac{\mu_s - \mu_m}{v_m} \le \dfrac{3}{8}\left(1-(\pi-3)\beta\right) \\ v_m\sqrt{\dfrac{4}{15}\left(1+\dfrac{4(1-\pi\beta)v_m}{3\pi(\mu_s-\mu_m)}\right)}, & \text{otherwise} \end{cases} \tag{2.44}$$

Here v_{b0} is given by (2.18). Thus, if the difference in static and dynamic friction is not too large, the strongest stick-slip oscillations occur when the excitation speed reaches the value separating stick-slip oscillations from pure-slip oscillations, $v_b = v_{b0}$. For larger friction differences the strongest oscillations occur at a smaller excitation speed, as given by the second expression in (2.44).

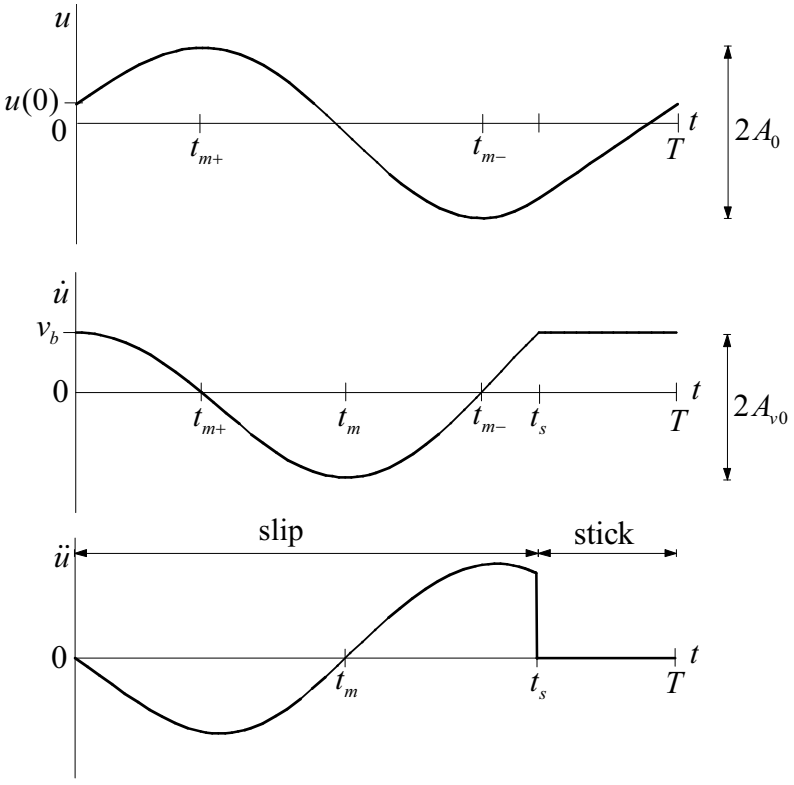

Fig. 2.7. Displacement, velocity, and acceleration during one cycle of stick-slip oscillation; definitions of displacement and velocity amplitudes A_0 and A_{v0}, switch time t_s, and the times t_{m-}, t_m, t_{m+} of maximum absolute displacements and velocities

Since the analysis assumes friction differences that are small (but finite), then the first case applies, so that we conclude that the strongest oscillations occur at $v_b = v_{b0}$:

$$A_0\big|_{v_b=v_{b0}} = A_{0,\max} = v_{b0} = \sqrt{\frac{4}{5}} v_m \sqrt{1 - \frac{4\beta v_m}{3(\mu_s - \mu_m)}} = A_{1,\max} \tag{2.45}$$

The last equality expresses that the predicted amplitude A_1 of pure-slip oscillations equals the predicted amplitude A_0 of stick-slip oscillations at the value of excitation speed separating these different kinds of motion. This exact continuity is neither obvious, nor required, since the two expressions were derived using approximate methods.

It can be noticed from (2.43) that at small values of the excitation speed, the stick-slip oscillation amplitude grows approximately linear with this speed:

$$A_0 \approx \left(1 - \pi\beta + \frac{3\pi}{4}\left(\frac{\mu_s - \mu_m}{v_m}\right)\right) v_b \quad \text{for} \quad v_b << 1 \tag{2.46}$$

Finally, since the velocity of the mass must change continuously with time, the maximum and minimum displacements of the mass must occur during the slip phase; they cannot occur during sticking, because displacements here increase linearly with time until the slip starts. Hence, the amplitude A_1, determining displacements during the slip phase, also determines the oscillation amplitude of the complete stick-slip oscillation.

Stick-Slip Base Frequency

Since at $t = T$ one cycle of slip and stick is completed, the base angular frequency of stick-slip oscillations is

$$\omega_{ss} = \frac{2\pi}{T} = \frac{2\pi v_b}{v_b t_s + u(0) - u(t_s)} \tag{2.47}$$

This frequency is generally somewhat lower than the linear natural frequency of the system and the frequency of pure slip oscillation.

2.1.5 Discussion of the Results

The type of stationary motion for the "mass-on-moving-belt" with friction law (2.4) depends on the speed of the belt.

Stick-slip oscillations with stationary amplitudes of displacement and velocity A_0 and A_{v0}, as given by (2.43) and (2.41), and base frequency ω_{ss} given by (2.47) occur for small excitation speeds of the belt (at these speeds the negative friction gradient is large). This type of oscillations exists until the belt's velocity reaches the critical value v_{b0} (2.18), (2.16). The belt's velocity corresponds to the maximal oscillations intensity, i.e. maximal displacement and velocity amplitudes.

If the belt's velocity exceeds this value (and the negative friction gradient decreases) stick-slip oscillations become impossible, because the mass doesn't get enough energy in order to catch up with the belt. The pure slip oscillations with stationary displacement and velocity amplitude A_1 as given by (2.17), and base frequency $\omega_{ps} = 1$ (equal to the linear natural frequency of the system) occur as long as the belt's speed doesn't exceed the second critical value v_{b1} (2.16). For some parameters it may happen that $v_{b0} = v_{b1}$, so that pure-slip oscillations cannot occur at all. This take place according to (2.16) if viscous damping is sufficiently large, or the friction forces or difference between static and kinetic friction is sufficiently small.

If the belt's velocity exceeds the second critical value $\left(v_b > v_{b1}\right)$, the static equilibrium at the position $x(t) = \overline{x}$ becomes stable. This situation corresponds to a steady state sliding of the mass.

Fig.2.8 shows the predicted variation of displacement amplitude with excitation speed for typical parameters. Here A_0 indicates the amplitude during stick-slip oscillations for and A_1 is the amplitude for pure slip oscillations. As it appears, when the excitation speed is increased from zero, stick-slip oscillations occur with increasing amplitude until pure-slip oscillations take over. These prevail only in a narrow range of belt velocities above which, oscillations cease and steady slip becomes the stable type of motion.

The simulations were performed for the following parameter values: $\beta = 0.05;\ v_m = 0.5;\ \mu_m = 0.25;\ \mu_s = 0.4$.

The correlation between the analytical predictions and numerical simulation (encircled points) is very good; closer correlation follows as the actual parameters values converge with the assumptions underlying the analytical expressions, i.e. as long as the difference in static and kinetic friction, the amount of viscous damping, and the amplitudes are not too large. It also appears that the predictions for the stick-slip amplitudes A_0 are much better than the low-order approximation (indicated in dashed line) given in [122].

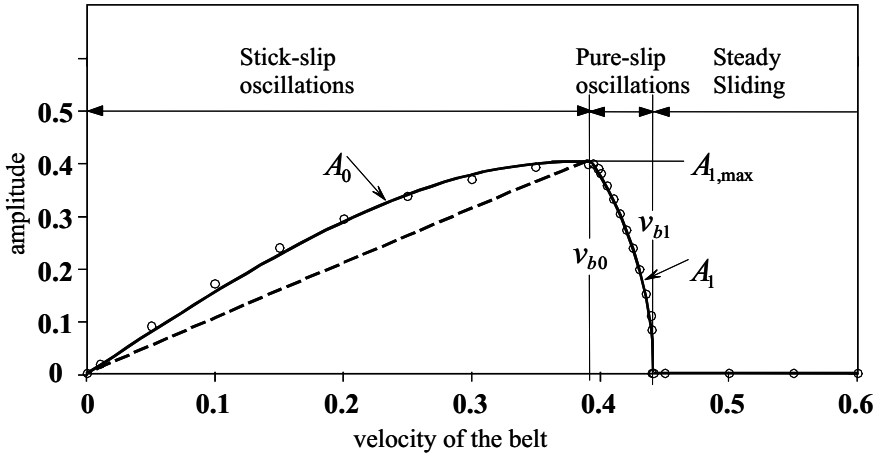

Fig. 2.8. Amplitude of stable periodic motions as a function of excitation speed v_b; the solid line corresponds to the analytical prediction; the dashed line corresponds to the analytical prediction from [122]; the dots correspond to numerical simulation results

Fig.2.9 shows the predicted variation of base frequency ω with excitation speed, showing a slight drop in frequency for the lower velocities. Seemingly, the correlation with numerical simulation is quite good; however, since the change in frequency is so small, the relative error in the prediction is somewhat larger than for the displacement and amplitude predictions. This is a consequence of the difficulty in predicting, with high accuracy, the end of the slip phase of a stick-slip cycle, based on an approximate expression for the displacements during that cycle.

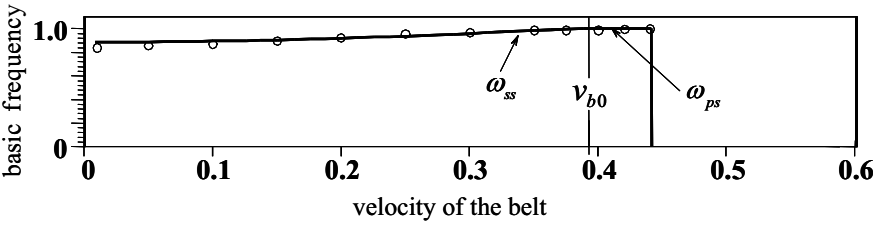

Fig. 2.9. Frequency of stable periodic motions as a function of excitation speed v_b; the solid line corresponds to the analytical prediction; the circles correspond to numerical simulation results

Fig. 2.10 illustrates how the quality of the analytical predictions changes as the assumptions underlying the analysis fails to hold true. The figure shows the variation of displacement amplitudes with excitation speed – just as Fig. 2.8, but for different levels of friction difference $(\mu_s - \mu_m)/v_m$ – and compare these to results of numerical simulation.

Solid lines correspond to the analytical prediction; dots (O, ×,□,⊕,✧) show the numerical simulation results for the following values of the parameter $(\mu_s - \mu_m)/v_m = (0.075;\ 0.099;\ 0.15;\ 0.3;\ 0.6)$ respectively. All the other parameters remain as before.

It is evident from the figure that the quality of the correlation deteriorates as $(\mu_s - \mu_m)/v_m$ increases from a value well within the assumed small size (0.075, circle symbols) to a value closer to unity (0.6, diamond symbols). The latter quite high level of friction gradient is an example where the second equation in (2.44) applies, e.g. the amplitude has a maximum at an excitation speed lower than the critical value separating stick-slip and pure slip oscillations. Higher order approximations will be required in order to improve on the accuracy for such large levels of friction gradient.

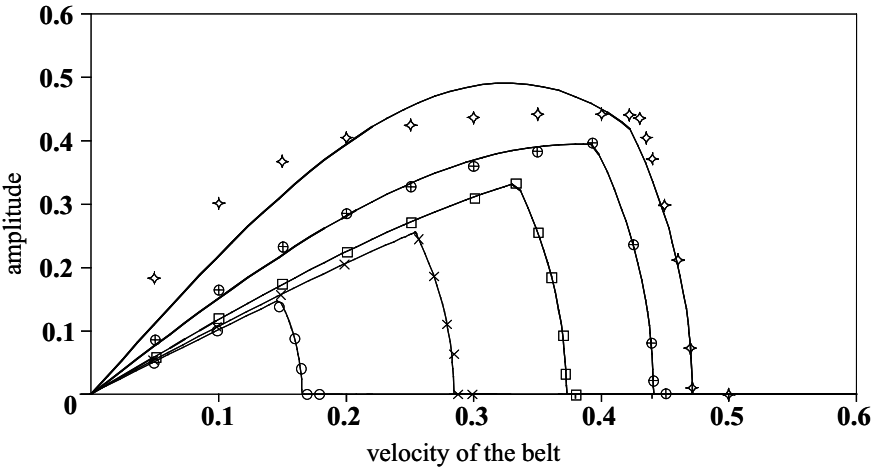

Fig. 2.10. Comparison between analytic and numeric prediction for different friction gradients

The figure also illustrates how the magnitude of oscillation amplitudes and the range of oscillation-producing velocities increase with the level of friction gradient. For the smaller values of the negative friction gradient, the relation between stick-slip amplitudes and excitation speed is very close to being linear, as predicted by (2.46).

2.1.6 Concluding Remarks

The example also shows how elaborated it can be, to apply "stitching" even to the simplest nonlinear systems (what we have done in case of the stick-slip oscillations). It is the only example of this approach in this book. In the next chapters we will use the "smoothing" methods like averaging and multiple scales in the

forms, which are modified in an appropriate manner, in order to make their use in discontinuous problems possible.

In the next section however, we will turn away from discontinuous problems in order to discuss another instability mechanism, caused by dry friction.

2.2 Friction Induced Flutter

When talking about flutter, we mean dynamic instability in a system with two or more degrees of freedom, which is mathematically caused by asymmetry of the stiffness matrix. This asymmetry (meaning nothing else but an energy source) can have different physical origins (see [85] for detailed discussion). Several simple examples are given in this section in order to illustrate how this kind of instability can be caused by dry friction. Let us start with mathematical basics.

2.2.1 Mathematical Basics of Flutter in a System with Two Degrees of Freedom

Consider the following linearized system:

$$\begin{aligned} &M\ddot{x}+B\dot{x}+Cx+Ax=0 \\ &M=\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix};\ B=\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix};\ A=\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \\ &\mathrm{C}=\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix},\quad c_{12}=c_{21} \ \Leftrightarrow\ C=C^T \end{aligned} \tag{2.48}$$

Here M is the matrix of inertia, B is the matrix of damping, C is the symmetric matrix of stiffness and A is the skew symmetric matrix corresponding to the non-conservative forces (energy source).

This system is obviously homogeneous and we are going to formulate the stability conditions for its trivial solution $x=0$. The corresponding characteristic equation is:

$$\begin{aligned} &\det\begin{vmatrix} m_1\lambda^2+b_1\lambda+c_{11} & c_{12}+a \\ c_{21}-a & m_2\lambda^2+b_2\lambda+c_{22} \end{vmatrix}=0 \quad \Leftrightarrow \\ &\left(m_1\lambda^2+b_1\lambda+c_{11}\right)\left(m_2\lambda^2+b_2\lambda+c_{22}\right)-\left(c_{12}+a\right)\left(c_{21}-a\right)=0 \end{aligned} \tag{2.49}$$

This is a fourth order algebraic equation with respect to the natural values λ:

$$m_1 m_2 \lambda^4 + \left(m_1 b_2 + m_2 b_1\right)\lambda^3 + \left(m_1 c_{22} + m_2 c_{11} + b_1 b_2\right)\lambda^2 + \left(c_{11} b_2 + c_{22} b_1\right)\lambda + c_{11} c_{22} + a^2 - {c_{12}}^2 = 0 \tag{2.50}$$

The Hurwitz' conditions [25] must be fulfilled, if all the natural values must have negative real parts:

$$a^2 > {c_{12}}^2 - c_{11} c_{22}$$
$$a^2 < {c_{12}}^2 - c_{11} c_{22} + \frac{\left(m_1 c_{22} + m_2 c_{11} + b_1 b_2\right)\left(c_{11} b_2 + c_{22} b_1\right)}{\left(m_1 b_2 + m_2 b_1\right)} - \frac{m_1 m_2 \left(c_{11} b_2 + c_{22} b_1\right)^2}{\left(m_1 b_2 + m_2 b_1\right)^2} \tag{2.51}$$

If the first inequality isn't fulfilled, the trivial solution increases exponentially without oscillations. If the second inequality isn't fulfilled, the trivial solution oscillates with the exponentially increasing amplitude. This type of instability is usually associated with flutter.

The inequalities (2.51) are sufficient conditions for the stability of the trivial solution. Sometimes it is sensible to analyze the system without damping (because it is usually not easy to estimate the real damping). In that case, the natural values can be calculated explicitly:

$$\lambda^2 = -\frac{m_1 c_{22} + m_2 c_{11}}{2 m_1 m_2} \pm \sqrt{\left(\frac{m_1 c_{22} + m_2 c_{11}}{2 m_1 m_2}\right)^2 - \frac{{c_{12}}^2 - c_{11} c_{22} + a^2}{m_1 m_2}} \tag{2.52}$$

Sufficient conditions for instability can be formulated in this case only, because it is impossible to make conclusions concerning stability of a nonlinear system when only a linearized system has been analyzed.

$$a^2 < {c_{12}}^2 - c_{11} c_{22}$$
$$a^2 > {c_{12}}^2 + \left(\frac{m_1 c_{22} - m_2 c_{11}}{2 m_1 m_2}\right)^2 \tag{2.53}$$

The system gets unstable if any of these inequalities is fulfilled. The types of instability are shown above.

2.2.2 Wobbling of an Elastically Supported Friction Disc

System shown in Fig. 2.11 is considered as the first example of the friction induced instability.

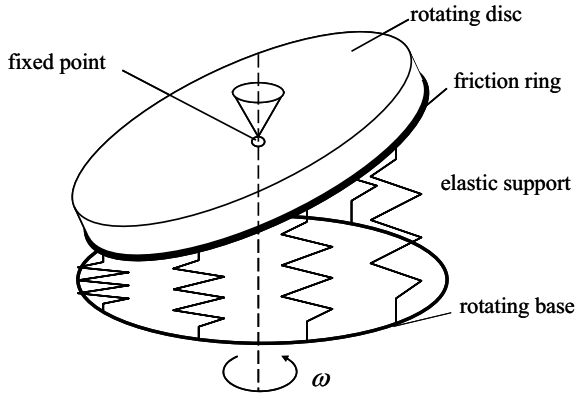

Fig. 2.11. Elastically supported friction disc

It consists of a homogeneous disc (radius R, thickness $2h$, equatorial moment of inertia J_e, polar moment of inertia J_p) which is fixed in its center of mass and can tilt in any direction. The disc doesn't rotate around its symmetry axis (otherwise we should take the gyroscopic terms into account, which would cause unnecessary complexities without any significant physical effect).

The disc is supported by a friction ring with the same radius R. The ring rotates together with the rigid base around the vertical axis. The connection between the ring and the base in vertical direction is elastic with the characteristic stiffness C. Inertia of the ring can be neglected in comparison with the inertia of the disc.

It is sensible to introduce the space frame (x, y, z) and the disc frame (ξ, η, ζ) in order to describe the motion of the disc (see Fig. 2.12 (a)).

The wobbling of the disc can be described by two generalized coordinates (α, β) (see Fig. 2.12 (b)).

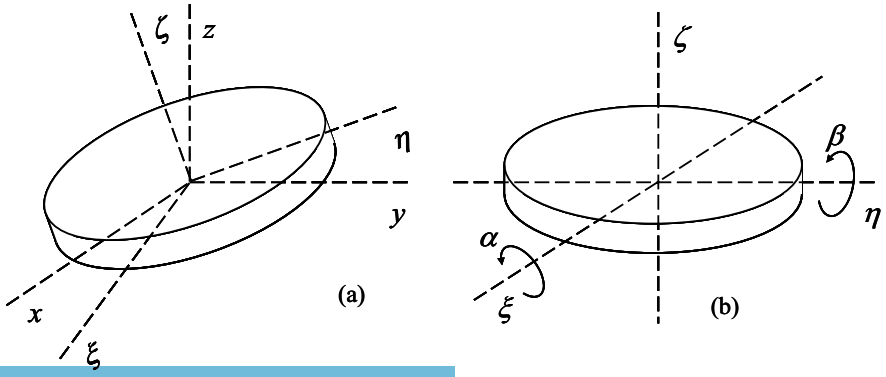

Fig. 2.12. Generalized coordinates describing the wobbling of the disc

The relationship between these two coordinate systems is as follows:

$$\begin{cases} x = \xi\cos\beta + \zeta\sin\beta \\ y = \xi\sin\alpha\sin\beta + \eta\cos\alpha - \zeta\sin\alpha\cos\beta \\ z = -\xi\cos\alpha\sin\beta + \eta\sin\alpha + \zeta\cos\alpha\cos\beta \end{cases} \tag{2.54}$$

This transformation can be linearized for small tilts:

$$\begin{cases} x = \xi + \zeta\beta \\ y = \eta - \zeta\alpha \\ z = -\xi\beta + \eta\alpha + \zeta \end{cases} \quad \Leftrightarrow \quad \begin{cases} \xi = x - z\beta \\ \eta = y + z\alpha \\ \zeta = x\beta - y\alpha + z \end{cases} \tag{2.55}$$

Our objective now is to calculate the reaction between the disc and the friction ring. We assume that the reaction of each element of the elastic layer is collinear with the vertical axis z and depends on the vertical coordinate of the corresponding point at the disc' surface only:

$$F_{ez} = F_{e0} - C\left(-\beta R\cos\varphi + \alpha R\sin\varphi\right) \tag{2.56}$$

Here φ is the angle along the friction ring. In the disc' frame we obtain:

$$\begin{cases} F_{e\xi}\left(\varphi\right) = F_{e0}\beta \\ F_{e\eta}\left(\varphi\right) = -F_{e0}\alpha \\ F_{e\zeta}\left(\varphi\right) = F_{e0} - CR\left(-\beta\cos\varphi + \alpha\sin\varphi\right) \end{cases} \tag{2.57}$$

The force component $F_{e\zeta} = N$ acts in the normal direction to the disc' surface and causes the friction force in each contact point (see Fig. 2.13).

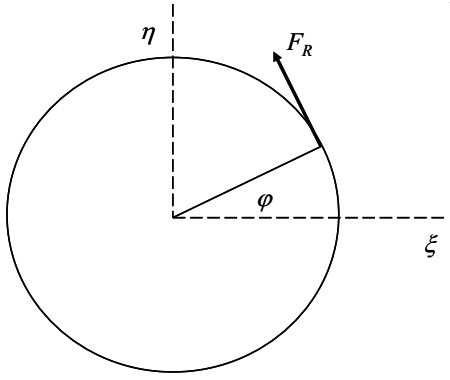

Fig. 2.13. Friction force in contact between the disc and the ring

Assuming the permanent slip in contact one can easily calculate the elementary friction force:

$$\begin{cases} F_{f\xi}(\varphi) = -F_f \sin\varphi = -\mu N \sin\varphi \\ F_{f\eta}(\varphi) = F_f \cos\varphi = \mu N \cos\varphi \\ F_{f\zeta}(\varphi) = 0 \end{cases} \tag{2.58}$$

Here μ is the friction coefficient (which we can assume constant in this section). Adding the force components (2.57) and (2.58) we obtain the full reaction force in each contact element between the disc and the ring:

$$\begin{cases} F_{\xi}(\varphi) = F_{e0}\beta - \mu \sin\varphi\left(F_{e0} - CR\left(-\beta\cos\varphi + \alpha\sin\varphi\right)\right) \\ F_{e\eta}(\varphi) = -F_{e0}\alpha + \mu\cos\varphi\left(F_{e0} - CR\left(-\beta\cos\varphi + \alpha\sin\varphi\right)\right) \\ F_{e\zeta}(\varphi) = F_{e0} - CR\left(-\beta\cos\varphi + \alpha\sin\varphi\right) \end{cases} \tag{2.59}$$

The corresponding elementary torque is as follows:

$$\begin{aligned} &\underline{\mathbf{M}}(\varphi) = \underline{\mathbf{r}}(\varphi) \times \underline{\mathbf{F}}(\varphi) \;\Rightarrow \\ &M_{\xi}(\varphi) = \eta(\varphi) F_{\zeta}(\varphi) - \zeta(\varphi) F_{\eta}(\varphi) \\ &M_{\eta}(\varphi) = \zeta(\varphi) F_{\xi}(\varphi) - \xi(\varphi) F_{\zeta}(\varphi) \end{aligned} \tag{2.60}$$

Substituting (2.59) into (2.60) and taking into account that at the contact line $\xi = R\cos\varphi,\ \eta = R\sin\varphi,\ \zeta = -h$ one obtains the explicit expressions for the elementary torques:

$$\begin{aligned} M_{\xi}(\varphi) &= R\sin\varphi\left(F_{e0} - CR\left(-\beta\cos\varphi + \alpha\sin\varphi\right)\right) \\ &\quad + h\left\{-F_{e0}\alpha + \mu\cos\varphi\left(F_{e0} - CR\left(-\beta\cos\varphi + \alpha\sin\varphi\right)\right)\right\} \\ M_{\eta}(\varphi) &= -h\left\{F_{e0}\beta - \mu\sin\varphi\left(F_{e0} - CR\left(-\beta\cos\varphi + \alpha\sin\varphi\right)\right)\right\} \\ &\quad - R\cos\varphi\left(F_{e0} - CR\left(-\beta\cos\varphi + \alpha\sin\varphi\right)\right) \end{aligned} \tag{2.61}$$

Integrating the elementary torques along the contact line one obtains the total torques responsible for the tilting of the disc:

$$M_\xi = \int_0^{2\pi} M_\xi(\varphi)\,d\varphi = -\left(\pi R^2 C + 2\pi F_{e0} h\right)\alpha + \pi\mu h R C \beta$$
$$M_\eta = \int_0^{2\pi} M_\eta(\varphi)\,d\varphi = -\pi\mu h R C \alpha - \left(\pi R^2 C + 2\pi F_{e0} h\right)\beta \tag{2.62}$$

Finally we obtain the equations of motion governing the tilting of the disc:

$$J_e\ddot{\alpha} + b\dot{\alpha} + \left(\pi R^2 C + 2\pi F_{e0} h\right)\alpha - \pi\mu h R C \beta = 0$$
$$J_e\ddot{\beta} + b\dot{\beta} + \pi\mu h R C \alpha + \left(\pi R^2 C + 2\pi F_{e0} h\right)\beta = 0 \tag{2.63}$$

The terms $b\dot{\alpha}$ and $b\dot{\beta}$ are introduced here in order to take the damping of the disc into account. Equations (2.63) have exactly the form that was discussed in section 2.2.1. Referring back to the form (2.48) we can notice that

$$c_{11} = c_{22} = \pi R^2 C + 2\pi F_{e0} h$$
$$c_{12} = c_{21} = 0 \tag{2.64}$$
$$a = -\pi\mu h R C$$

In other words the tilting of the disc with respect to both axes is coupled through the skew symmetric matrix caused by dry friction. This mechanism is illustrated in Fig. 2.14 and can be physically explained as follows.

Let us assume the disc tilts around the ξ-axes. The springs on the left side of the Fig. 2.14 are more highly stressed than those on the right side as a consequence. The corresponding reaction forces on the left side are larger. They cause the restitution torque, which is mathematically expressed through the terms at the main diagonal of the stiffness matrix. The relative motion between the disc and the friction layer causes friction forces tilting the disc around the η-axes.

The friction induced tilting torque acts in opposite direction on the left and on the right side of the disc. The resultant torque would be equal to zero, if the spring forces were uniform. In our situation however, they are non homogeneously distributed and the torque acting on the left side of the disc is larger than the torque acting on its right side. As a result a non compensated torque occurs. It tilts the disc around the η-axes. This fact is mathematically expressed by the skew symmetric terms in the stiffness matrix.

Now we could repeat the analysis assuming the disc is tilted around the new axis. The result would be the same. Tilting around any axis causes tilting around another axis, which is perpendicular to the first one. The disc starts wobbling.

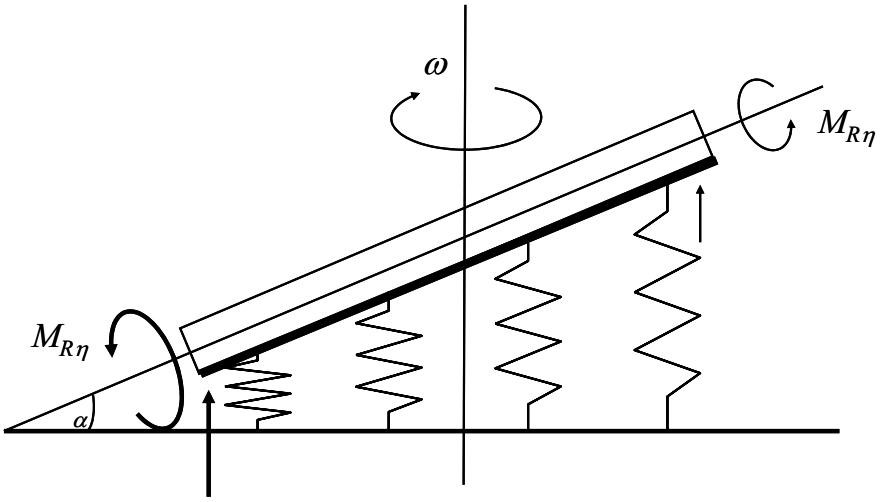

Fig. 2.14. Tilting torque due to non homogeneous friction forces

The stability conditions (2.51) take in our case the following simple form:

$$a^2 > -{c_{11}}^2$$
$$a^2 < b^2 \frac{c_{11}}{J} \tag{2.65}$$

The first inequality is always fulfilled. The second one can be transformed to the following explicit form:

$$\mu < b \frac{\sqrt{R^2 C + 2F_{e0} h}}{hRC\sqrt{\pi J}} \tag{2.66}$$

This result shows that a tilting disc on the elastic friction layer without damping would be always unstable! This result is relevant for oscillations of friction clutches ("eek noise") and layers. It can be easily generalized for a rotating disc. The gyroscopic terms should be taken into account in that case. The stability condition, determining the maximal acceptable friction coefficient, depends then on the disc's rotation speed.

2.2.3 On the Unstable Behavior of an Asymmetrically Supported Disc (Brake Squeal)

An elementary model of brake squeal is discussed in this section as the second example. This model was suggested by Peter Hagedorn in a personal communication, who kindly allowed including this section into the book.

Consider the system shown in Fig. 2.15.

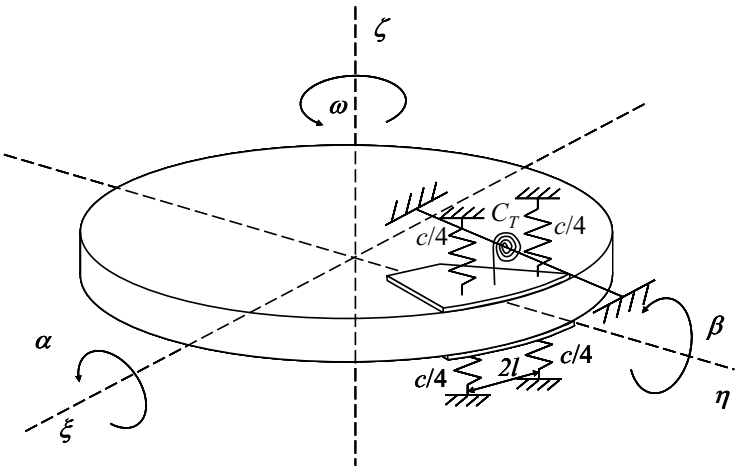

Fig. 2.15. A rotating disc with an asymmetric elastic support

It consists of a rigid rotating disc pressed between two friction pads. These pads are supported elastically with respect to both in axial displacement and tilting around the η-axis. The axial springs are preloaded with the same force N_0. The forces acting on the disc are shown in Fig. 2.16.

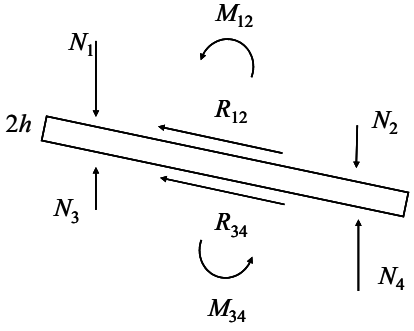

Fig. 2.16. The forces and torques acting on the disc

These forces can be calculated as follows:

$$
\begin{aligned}
&N_1 = N_0 + \frac{1}{4}cr\alpha - \frac{1}{4}cl\beta;\; N_2 = N_0 + \frac{1}{4}cr\alpha + \frac{1}{4}cl\beta \\
&N_3 = N_0 - \frac{1}{4}cr\alpha - \frac{1}{4}cl\beta;\; N_4 = N_0 - \frac{1}{4}cr\alpha + \frac{1}{4}cl\beta \\
&R_{12} = \mu\left(N_1 + N_2\right) = 2\mu N_0 + \frac{1}{2}\mu cr\alpha \\
&R_{34} = \mu\left(N_3 + N_4\right) = 2\mu N_0 - \frac{1}{2}\mu cr\alpha \\
&M_{12} = -C_T\beta \\
&M_{34} = 0
\end{aligned}
\tag{2.67}
$$

Now we can calculate the total torques acting on the disc (we neglect the gyroscopic terms as before):

$$
\begin{aligned}
M_\xi &= -\left(N_1 + N_2 - N_3 - N_4 + R_{12}\beta + R_{34}\beta\right)r \\
&= -cr^2\alpha - 4\mu N_0 r\beta \\
M_\eta &= M_{12} + \left(N_1 - N_2 - N_3 + N_4\right)l + \left(R_{12} - R_{34}\right)h \\
&= C_T\beta + \mu crh\alpha
\end{aligned}
\tag{2.68}
$$

The corresponding linearized equations of motion are as follows:

$$
\begin{aligned}
&J\ddot{\alpha} + cr^2\alpha + 4\mu N_0 r\beta = 0 \\
&J\ddot{\beta} - \mu crh\alpha + C_T\beta = 0
\end{aligned}
\tag{2.69}
$$

These equations have a well known structure. The two degrees of freedom are coupled non-symmetrically through the dry friction, which is the energy source amplifying the oscillations. Referring back to the general equations (2.48) one can find the following relations:

$$
\begin{aligned}
&c_{11} = cr^2;\; c_{22} = C_T;\; c_{12} = c_{21} = \mu\frac{r}{2}\left(4N_0 - ch\right) \\
&a = \mu\frac{r}{2}\left(4N_0 + ch\right)
\end{aligned}
\tag{2.70}
$$

The corresponding inequalities (each of them is sufficient for the instability) (2.53) are:

$$4\mu^2 N_0 h < -C_T; \quad \mu > \frac{\left|C_T - cr^2\right|}{4Jr\sqrt{N_0 ch}} \tag{2.71}$$

The first inequality is never fulfilled. The second one can be fulfilled if the friction coefficient is sufficiently large. Then the oscillating instability occurs.

The mechanism described is strongly simplified. The realistic model of the brake squeal does not consider the motions of the friction disc as a rigid body. It must take (according to P. Hagedorn) the bending modes of the disc into account, but the logic described and the mechanism of instability remains the same.

2.2.4 Conclusions

The friction induced flutter is an important source of instability in systems with dry friction. Most of books and articles devoted to friction induced vibrations concentrate on the negative friction gradient as the main source of instability. Practical experience has convinced the author that the mechanism described in this section is not less important, but unfortunately almost unknown by mechanical engineers. That's why it is considered in this chapter even though the discussion here is limited to the elementary analysis of stability and the effects of nonlinearity are not taken into account.

2.3 Vibration Induced Displacement. Averaging in Discontinuous Systems

Let us move on to the last group of problems connected with dry friction we are going to discuss in this chapter. We are going to discuss the so called "*vibration induced displacement*" or in other words the phenomenon transforming symmetric oscillations (excitation) into directed motion of the system. (Sometimes it is called "vibration induced transportation" or "vibration induced translation".) This phenomenon is used in numerous machines in chemical and civil engineering like vibrating conveyors, screens and other machines transporting bulk materials. Hundreds different designs of these machines are used in practice. The phenomenon, however, is not only of practical, but mostly of general importance. It demonstrates the ability of vibration to use the asymmetry of the system, in order to steer the input energy into some preferable direction. Besides that, it gives an excellent opportunity to demonstrate the peculiarities of discontinuous systems and introduce the appropriate averaging procedure. Let us start with an elementary example.

2.3.1 A Simple Example of the Vibration Induced Displacement

A simple example of a slipping system is discussed in this section, to show typical problems which can occur in the analysis of discontinuous systems. Consider a model of vibration induced displacement shown in Fig. 2.17. This system consists of a mass on a friction plate. Friction coefficients are taken to be different if the mass moves forwards or backwards. The mass is excited by a harmonically oscillating horizontal force.

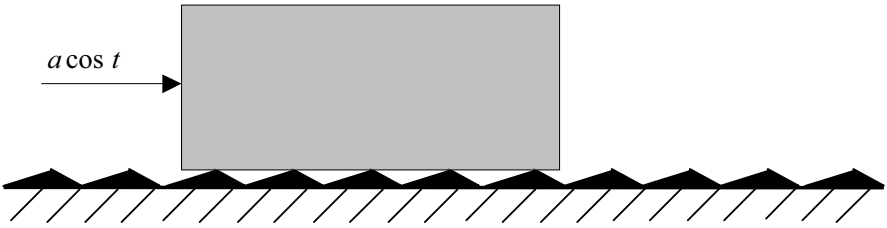

Fig. 2.17. The basic example of the vibration induced displacement

The equation of motion for the mass can be written as follows:

$$\begin{aligned} m\ddot{x} &= a\cos t - F_{friction} \\ F_{friction} &= \mu_f mgE\left(\dot{x}\right) - \mu_b mgE\left(-\dot{x}\right) \end{aligned} \tag{2.72}$$

Here x is the coordinate of the mass on the plane, a is the excitation' amplitude, g is the gravity' acceleration. Constants μ_f and μ_b are the friction coefficients forwards and backwards, which are assumed to be different. The one step function $E\left(\dot{x}\right)$ is determined as follows:

$$E\left(\dot{x}\right) = \begin{cases} 1 \text{ if } \dot{x} > 0 \\ \frac{1}{2} \text{ if } \dot{x} = 0 \\ 0 \text{ if } \dot{x} < 0 \end{cases} \tag{2.73}$$

The one step function and the asymmetric friction law (2.72) are illustrated in Fig. 2.18.

The assumption about the asymmetry of the friction force is not as unusual as it may seem. The majority of the surfaces has some structure due to production process (like metal cutting) and is clearly anisotropic.

Equation (2.72) does not contain x. So we are not interested in the position of the mass but only in its velocity. Introducing the average friction force and the asymmetry of friction we can rewrite (2.72) as follows:

$$\dot{x} = v$$
$$\dot{v} = \tilde{a}\cos t - \bar{\mu}\varepsilon\tilde{a}\,\mathrm{sgn}\, v - \tilde{\mu}\varepsilon\tilde{a}$$
$$\tilde{a} = \frac{a}{m};\ \bar{\mu}\varepsilon = \frac{g}{2\tilde{a}}\left(\mu_f + \mu_b\right);\ \tilde{\mu}\varepsilon = \frac{g}{2\tilde{a}}\left(\mu_f - \mu_b\right) \tag{2.74}$$
$$\mathrm{sgn}\, v = E\left(v\right) - E\left(-v\right)$$

The introduced formal small parameter ε indicates the assumption that the friction force is small.

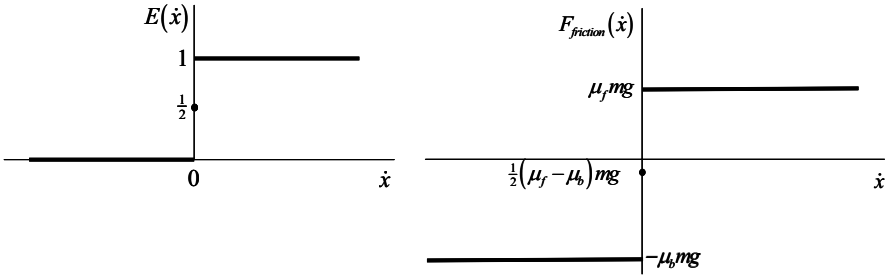

Fig. 2.18. The one step function and the asymmetric friction law

The basic idea for transformation of a perturbed system to a form suitable for averaging is to consider the unperturbed system. In out case it is:

$$\dot{v}_0 = \tilde{a}\cos t \tag{2.75}$$

Its solution $v_0 = \tilde{a}\sin t + const$ can be considered as a transformation for the equation (2.74):

$$v = \tilde{a}\sin t + u\tilde{a} \tag{2.76}$$

The new unknown function u is governed by the following equation:

$$\dot{u} = -\bar{\mu}\varepsilon\,\mathrm{sgn}\left(u + \sin t\right) - \tilde{\mu}\varepsilon \tag{2.77}$$

It seems to be an equation in the standard form for averaging. Unfortunately this is not the case, even though the right hand side of this equation is proportional to the small parameter, limited and periodic. The problem is that the sgn-function is discontinuous and it does not fulfill the Lipschitz-condition as a consequence. However this system has an important peculiarity, which enables generalized averaging for these specific types of discontinuities. The argument of the discontinuous function is a sum of a slowly changing function (in our case it

is u, $\dot{u} = O(\varepsilon)$) and a fast periodic function (in our case it is $\sin t$, $(\sin t)^{\|} = \cos t = O(1)$).

2.3.2 Mathematical Basics for the First Order Averaging of the Constant Order Discontinuous Regimes

The following theorem can be formulated and proved for the systems of this type.

Theorem 2.1
Consider an initial value problem

$$\dot{x} = \varepsilon Z(x,t) E\left(g(t) + f(x)\right),\ x(0) = x_0 \tag{2.78}$$

Alongside (2.78) *consider the corresponding averaged system:*

$$\begin{aligned} &\dot{\xi} = \varepsilon \zeta(\xi),\ \xi(0) = x_0 \\ &\zeta(\xi) = \left\langle Z(\xi,t) E\left(g(t) + f(\xi)\right)\right\rangle \end{aligned} \tag{2.79}$$

Here E *is the one step function.*
Suppose:

1. $Z: R^{n+1} \to R^n$ *is a bounded, Lipschitz-continuous in* x, T*-periodic in* t *function for* $x \in D \subset R^n$, $t \in [0,\infty)$, $\varepsilon \in (0, \varepsilon_0]$; *(these requirements are usual for averaging).*

2. $f: R^n \to W_f \subset R^1$ *is a bounded differentiable function in* x *on* D. *(There is also nothing unusual in this requirement, except the fact that we have introduced the set* W_f, *which is the set, from which the function* f *takes its values.)*

3. $g: R^1 \to W_g \subset R^1$ *is a continuously differentiable function of time. (The set* W_g *is introduced here.)*

The next two conditions are important:

4. $W_f \subset W_g$ *(The set* W_f *is a subset of* W_g*)*

5. Consider an equation

$$g(t) = const \in W_f \tag{2.80}$$

We suppose that this equation for any constant value on the right hand side has a finite number of isolated solutions inside of any period of the function $g: t \in [0,T)$ *:*

$$t = t_{i0},\ i = 1,2,\ldots,m \tag{2.81}$$

The requirement that these solutions have to be isolated, means that the first derivative of the function g *in these points is not equal to zero:*

$$\left|g'\left(t_{i0}\right)\right| \geq G > 0 \tag{2.82}$$

This condition is of the principal significance for this theorem. It is graphically illustrated in Fig. 2.19. Qualitatively it means that the time during which our system remains in the vicinity of the discontinuity is short. It is the case if the function g changes much faster that the function f. This requirement could be broken if the zero of the function $f(x)+g(t)$ corresponding to the discontinuity would meet a maximum or a minimum of the function g. This possibility is eliminated by the fifth requirement (2.80) - (2.82).

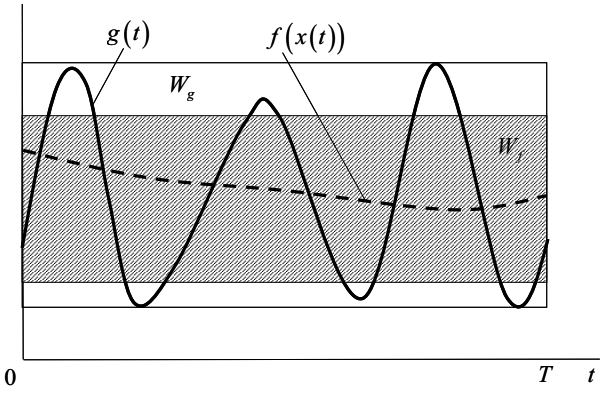

Fig. 2.19. Requirements to the function $g(t)$

6. All the constants do not depend on the small parameter. The solution to the averaged system (2.79) *belongs to the interior subset of* D *on the time scale* $1/\varepsilon$ *.*

Under these conditions the solutions to (2.78) *and* (2.79) *are asymptotically close to each other, i.e.*

$$\left\|x(t)-\xi(t)\right\| \leq c_1 \varepsilon e^{c_2 \varepsilon t},\ c_1 = O(1),\ c_2 = O(1) \tag{2.83}$$

In other words, under these conditions the system (2.78) can be averaged (with the first order accuracy) as if it were continuous.

The proof of this theorem can be found in the Appendix IV.

This theorem validates averaging for the case that the system does not remain in the vicinity of the discontinuity. For systems with dry friction, it means pure slip motion is occurring. It is natural to call motions of this type the "*constant order discontinuous regimes*". This name underlines their difference from the "*variable order discontinuous regimes*" like stick-slip oscillations in the systems with dry friction.

Let us return back to our example of the vibration induced displacement and look how the described procedure works there.

2.3.3 The Elementary Example of the Vibration Induced Displacement. The First Order Approximation

The equation (2.77) satisfies all the conditions of the theorem 2.1 if we consider the case $-1 < u < 1$. Then we can directly average its right hand side and obtain the equation of the first order approximation. Firstly we have to calculate the average:

$$\left\langle \operatorname{sgn}\left(u+\sin t\right)\right\rangle = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{sgn}\left(u+\sin t\right) dt = \frac{2}{\pi}\arcsin u \tag{2.84}$$

The equation of the first order approximation is:

$$\dot{u}_1 = -\frac{2}{\pi}\bar{\mu}\varepsilon \arcsin u_1 - \tilde{\mu}\varepsilon \tag{2.85}$$

Here index 1 denotes the averaged functions of the first order approximation. The stationary solution to (2.85) can be found by setting its right hand side to zero:

$$u_{1\_st} = -\sin\left(\frac{\pi\tilde{\mu}}{2\bar{\mu}}\right) \tag{2.86}$$

The accuracy of this solution can be easily checked by numerical simulation. The comparison is shown in Fig. 2.20.

The comparison was performed for the following values of the parameters: $\bar{\mu}\varepsilon = 0.1;\ \tilde{\mu}\varepsilon = -0.05;\ \tilde{a} = 1$. The accuracy of the approximate solution seems to be quite acceptable.

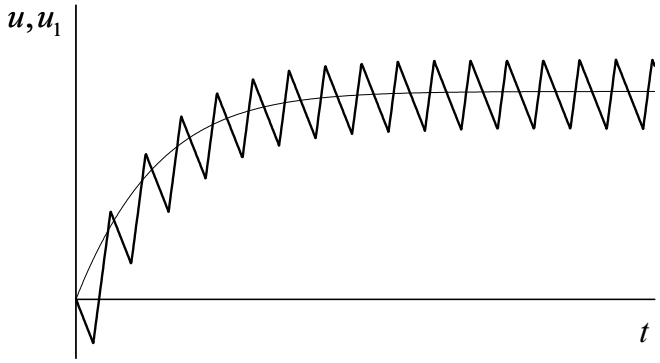

Fig. 2.20. Comparison between numerical simulation and analytic prediction; the thick saw-like line corresponds to the results of numerical simulations, the smooth thin line corresponds to the solution of the averaged initial value problem (2.85)

2.3.4 Discussion of the Results

The result achieved is physically obvious. The oscillating mass on the surface with anisotropic friction moves on the average in the direction of the smaller friction.

The same practical effect can be achieved if we subject the mass on the surface with isotropic friction to elliptic excitation (which is the overlapping of synchronous vertical and horizontal excitations). This example is discussed in the chapter 6. Here we will give only the main results.

Consider a system shown in Fig. 2.21.

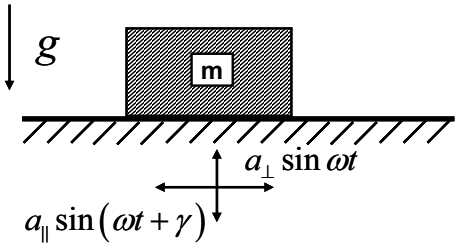

Fig. 2.21. A mass on the elliptically excited plane

The equations governing the motion of the mass can be written as follows:

$$m\ddot{x} = -N\mu\left(\dot{x}\right) + ma_{\perp}\omega^2 \sin\omega t$$
$$N = mg - ma_{\parallel}\omega^2 \sin\left(\omega t + \gamma\right) \tag{2.87}$$

These equations express the fact that the friction force is proportional to the normal reaction force. In our case, the normal force is influenced by the vertical acceleration of the plane. Thus the asymmetry of the excitation is the reason of the translation. It presses the mass towards the plane as it moves backwards and reduces the pressure in the phases, when the mass moves forwards. These influences result in a slow steady state translation overlapped by small fast vibrations (cf. Fig. 2.20). There are many machines based on the principle we have discussed above. First of all vibrating conveyors and sieves are used in the processing of bulk materials.

Figure 2.22 shows how the average velocity of the translation depends on the phase difference between the vertical and the horizontal excitations. The numerical simulations are compared with analytic prediction obtained in chapter 6.

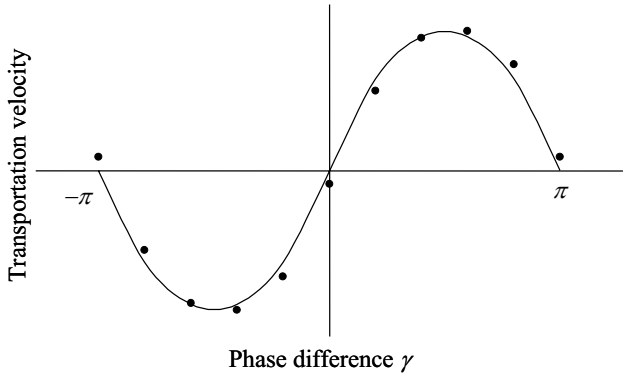

Fig. 2.22. Transportation of the mass on the elliptically vibrating surface; the smooth line corresponds to the results of numerical simulation, the dots correspond to the analytic prediction

2.3.5 Conclusions

Averaging can be used directly, in order to analyze the constant order regimes in discontinuous systems with the accuracy of the first order approximation. This approach is very effective in slipping systems with dry friction and enables us to describe the useful mechanisms of the vibration induced displacement. In the next section we are going to discuss an advanced case, which requires analyzing the second order approximation. It will lead us to a more detailed insight into the peculiarities of discontinuous systems.

2.4 Vibration Induced Displacement of a Resonant Friction Slider

2.4.1 Problem Description

Following [40], the problem of predicting steady drifting motions for a rigid sliding body, sandwiched between two friction layers, as induced by an internal resonator driven at high frequency is considered in this section. Fig.2.23 shows the system under consideration: A rigid slider with an imbedded oscillator. The oscillator is driven near a high-frequency resonance. Two parallel surfaces of the slider are pre-loaded against friction layers having different frictional properties. This creates friction forces that depend asymmetrically on loads normal to the friction layers. With the internal resonator aligned neither normal nor parallel to the friction layers, the inertia forces of the resonator can cause the slider to move in one direction with a small overlay of high-frequency vibrations. Thus, a part of the energy supplied to small but fast oscillations of the resonator will be converted into large and comparably slow trenslation of the slider.

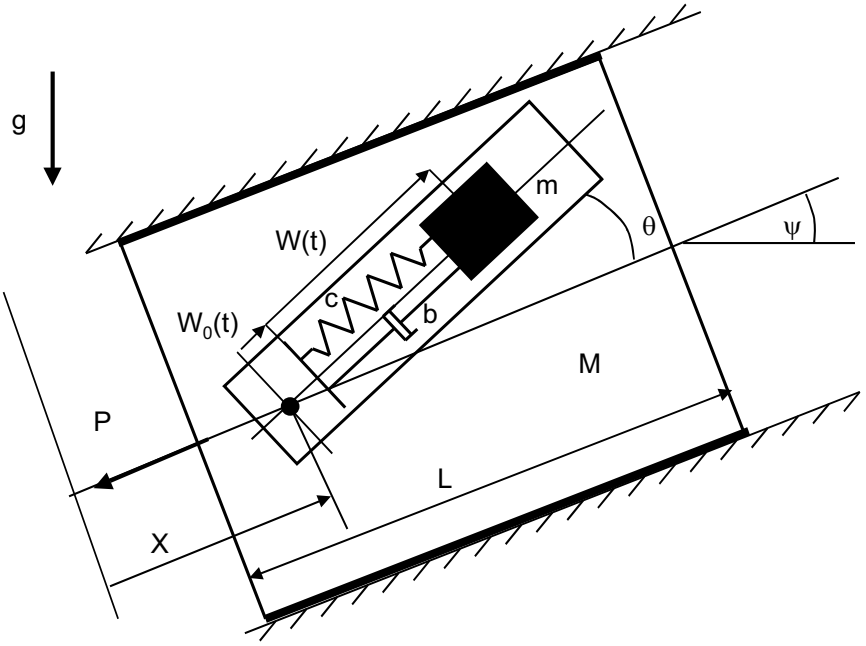

Fig. 2.23. The resonant friction slider

The system shown in Fig. 2.23 consists of a slider, a pair of stators, and a resonator. The slider is considered a rigid body of length L and mass M. At time t

its centre of mass G is located a distance $X(t)$ from a fixed point in space. Rotational motion is prevented, so rotary inertia is ignored. A force P acts against the slider, representing some external loading of the system.

The stators provide a no-clearance guide way for the slider, fixed in space and tilted an angle ψ from horizontal in the gravity field g.

The resonator is considered a linear 1-dof oscillator, with a mass m at instantaneous position $W(t)$, stiffness k, coefficient of viscous damping c, and mounted with a tilt θ inside the slider. The resonator base is driven kinematically at small amplitude and high frequency, close to its natural frequency, i.e. close to resonance. Thus $W_0(t) = A\sin\omega t$ is the externally controlled displacement of the resonator base, where $A << L$ is the drive amplitude, and ω is the near-resonant drive frequency: $\omega \approx \sqrt{k/m} >> \sqrt{g/L}$. Physically the resonator might be realized by a conventional ultrasonic transducer clamped at one end inside the slider, and driven near resonance by piezo-electric ceramics (at amplitude about 1 μm and frequency about 30 kHz).

The contact between the slider and the stators is governed by dry friction with different friction coefficients μ_1 and μ_2 for the lower and upper contact surface, respectively. The friction force at surface i is then $F_i = \mu_i N_i \operatorname{sgn}\dot{X}$, where $N_i > 0$ is the compressive force normal to the surface. The slider is preloaded by a force N_0 between the stators, so that $N_1 = N_2 = N_0$ when the system is at rest and gravity does not contribute to the normal force ($\psi = \pi/2$). This pre-load is assumed to be sufficiently large to maintain steady contact at both slider-stator interfaces during operation.

2.4.2 Equations of Motion

We can derive the equations of motion in the Lagrange's form if we calculate the kinetic and potential energies, the Rayleigh's dissipative function and the generalized forces as follows:

$$\tilde{T}=\frac{1}{2}\left(M+m\right)\dot{X}^2+m\dot{X}\left(\dot{W}_0+\dot{W}\right)\cos\theta+\frac{1}{2}m\left(\dot{W}_0+\dot{W}\right)^2$$

$$\tilde{V}=\left(M+m\right)gX\sin\psi+mg\left(W_0+W\right)\sin\left(\psi+\theta\right)+\frac{1}{2}kW^2 \qquad (2.88)$$

$$\tilde{Q}_X=-(P+F_1+F_2);\quad \tilde{Q}_W=0;\quad \tilde{D}=\frac{1}{2}c\dot{W}^2$$

Here F_1, F_2 are the friction forces described above. The Lagrange's equations have the following form:

$$\frac{d}{dt}\left(\frac{\partial\tilde{L}}{\partial\dot{X}}\right)-\frac{\partial\tilde{L}}{\partial X}+\frac{\partial\tilde{D}}{\partial\dot{X}}=\tilde{Q}_X;\quad \frac{d}{dt}\left(\frac{\partial\tilde{L}}{\partial\dot{W}}\right)-\frac{\partial\tilde{L}}{\partial W}+\frac{\partial\tilde{D}}{\partial\dot{W}}=\tilde{Q}_W \qquad (2.89)$$

$$\tilde{L}\equiv\tilde{T}-\tilde{V}$$

Substituting (2.88) into (2.89) one obtains the equations governing the motion of the slider with the internal resonator:

$$\begin{aligned}\left(M+m\right)\ddot{X}+\left(F_1+F_2\right)+m\left(\ddot{W}_0+\ddot{W}\right)\cos\theta&\\ =-P-\left(M+m\right)g\sin\psi&\qquad (2.90)\\ m\ddot{W}+c\dot{W}+kW+m\ddot{X}\cos\theta=-m\ddot{W}_0-mg\sin\left(\psi+\theta\right)&\end{aligned}$$

We need to compute the normal forces N_1 and N_2 at the friction layers, in order to determine the friction forces F_1+F_2. These will change in time because they are determined by the inertia forces transmitted from the vibrating internal resonator. If the two friction layers have identical normal stiffness, any increase ΔN in the compressive normal load exerted by the slider on one surface will be balanced by an equal decrease in the load on the other surface, until a clearance occurs as soon as one of the normal forces gets equal to zero.

In this study we consider the case $\left|\Delta N\right|<N_0$ only, which means: The pre-load is sufficiently large to prevent clearance while the device is operating. For this case one obtains

$$\begin{aligned}&N_1=N_0+\Delta N;\ N_2=N_0-\Delta N\\ &F_1+F_2=\left(\mu_1N_1+\mu_2N_1\right)\operatorname{sgn}\dot{X}=\\ &\qquad=\left(\mu_1+\mu_2\right)N_0\operatorname{sgn}\dot{X}+\left(\mu_1-\mu_2\right)\Delta N\operatorname{sgn}\dot{X}\end{aligned} \qquad (2.91)$$

ΔN can be obtained if we balance the forces normal to the stators:

$$\Delta N = \frac{1}{2} m\left(\ddot{W}_0 + \ddot{W}\right)\sin\theta + \frac{1}{2}\left(M+m\right) g\cos\psi \tag{2.92}$$

The condition ensuring compressive slider loading at all times is

$$N_0 > \max_t \left\{\left|\Delta N\left(t\right)\right|\right\} \tag{2.93}$$

Let us assume that the external excitation is a harmonic one:

$$W_0 = A\sin(\omega t) \tag{2.94}$$

Firstly we are going to bring the problem to the non-dimensional form. We introduce the following frequencies, non-dimensional variables and parameters:

$$\begin{gathered} \omega_0 = \sqrt{\frac{k}{m}};\ \ \lambda = \sqrt{\frac{g}{l}};\ \ \ \varepsilon_f = \frac{\lambda^2}{\omega_0{}^2} \\ x = \frac{X}{L};\ \ \ w = \frac{W}{L} + \varepsilon_f \sin\left(\psi + \theta\right) \end{gathered} \tag{2.95}$$

$$\begin{gathered} \beta = \frac{c}{2\sqrt{km}\sqrt{1-\alpha\cos^2\theta}};\alpha = \frac{m}{m+M};\ p = \frac{P}{\left(m+M\right)g} \\ \Omega = \frac{\omega}{\omega_0}\sqrt{1-\alpha\cos^2\theta};a = \frac{A}{L};n_0 = \frac{2N_0}{mL\omega_0{}^2};\Delta n = \frac{2\Delta N}{mL\omega_0{}^2} \\ \tau = \frac{\omega_0 t}{\sqrt{1-\alpha\cos^2\theta}};\ \left(\ \right)' = \frac{d\left(\ \right)}{d\tau} \end{gathered} \tag{2.96}$$

Now we can rewrite the equations (2.91), (2.92) as follows:

$$\begin{gathered} x'' + \alpha\left(w'' - a\Omega^2\sin\Omega\tau\right)\cos\theta = \\ -\left\{\varepsilon_f\left(p+\sin\psi\right) + \alpha\bar{\mu}\left(x'\right)n_0 + \alpha\mu_\Delta\left(x'\right)\Delta n\right\}\left(1-\alpha\cos^2\theta\right) \\ w'' - a\Omega^2\sin\Omega\tau + \left(2\beta w' + w\right)\left(1-\alpha\cos^2\theta\right) + x''\cos\theta = 0 \end{gathered} \tag{2.97}$$

The following signs are used here:

$$\Delta n = \frac{w'' - a\Omega^2 \sin\Omega\tau}{1-\alpha\cos^2\theta}\sin\theta + \frac{\varepsilon_f}{\alpha}\cos\psi; \qquad n_0 \geq \max\left(\left|\Delta n\right|\right)$$
$$\bar{\mu}(x') = \bar{\bar{\mu}}\,\mathrm{sgn}\left(x'\right); \qquad \bar{\bar{\mu}} = \frac{1}{2}\left(\mu_1 + \mu_2\right) \tag{2.98}$$
$$\mu_\Delta(x') = \bar{\tilde{\mu}}\,\mathrm{sgn}\left(x'\right); \quad \bar{\tilde{\mu}} = \frac{1}{2}\left(\mu_1 - \mu_2\right)$$

Note that Ω denotes the non-dimensional drive frequency and a is the amplitude of the internal resonator, which has natural frequency equal to unity and damping ratio β. $\alpha \in [0,1]$ denotes the relative mass of the resonator, w the position of the resonator' mass relative to its static equilibrium, $\bar{\bar{\mu}}$ the average friction coefficient for the two friction layers, and $\bar{\tilde{\mu}}$ expresses the difference (asymmetry) in friction coefficients, n_0 is the pre-load of the slider, and p the external load.

In the next section we illustrate, which types of the system's behavior can be obtained by numerical simulation, and then consider obtaining approximate analytic solutions.

2.4.3 Illustration to System's Behavior

Numerical solutions to (2.97) were obtained using the commercial software package ITI-SIM. (It is the case for almost all numeric simulations in this book). Fig. 2.24 shows three responses for the slider motion $x(\tau)$, when the excitation amplitude is, respectively, $a = 0.05$ (bottom curve), $a = 0.02$ (middle), and $a = 0.005$ (top curve), and other parameters as given in the figure legend. The excitation is resonant for all curves, i.e. $\omega = \omega_0$; $\Omega = 1$. Also, the level of supplied input energy is the same, i.e. the product $a\omega$ is constant (this corresponds to choosing $\varepsilon_f = a^2$), while the pre-load n_0 in each case has been chosen sufficiently large to prevent slider-stator clearance. The results are displayed on a time scale $a\tau$, which is proportional to physical time t.

As appears the slider climbs against gravity ($\psi = 10^\circ$), moving at constant average velocity with a small overlay of rapid oscillations. The average velocity is of similar order of magnitude for all three cases, whereas the amplitude of the overlaid oscillations decreases with the decreasing excitation's amplitude (and correspondingly increasing excitation frequency). For $a = 0.005$ the high frequency

overlay is hardly noticeable, and the slider appears to move smoothly in one direction.

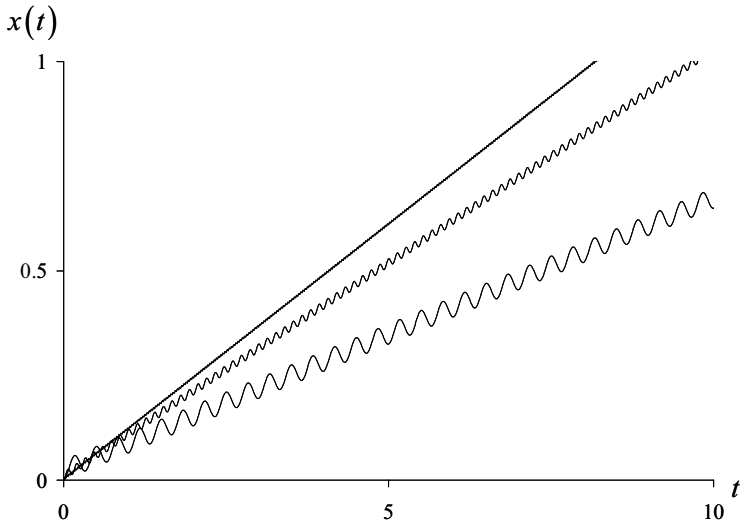

Fig. 2.24. Slider position obtained for the following parameter values: $p = 0$, $\psi = 10^{\circ}$, $\theta = 45^{\circ}$, $\mu_{1k} = 0.25$, $\mu_{2k} = 0.05$, $\beta = 0.05$, $\alpha = 0.1$

The physical phenomenon causing average translation of the slider is simple: When $\theta \neq \pi/2$ (*cf.* Fig. 2.23.) and friction is not too large, inertia forces from the vibrating resonator cause the slider to oscillate back and forth between the stators. When in addition $\theta \neq 0$, the inertia forces also create oscillating normal forces on the slider surfaces, and thus oscillating frictional forces parallel to the surfaces. If the surfaces differ in frictional properties $\mu_{\Delta} \neq 0$, then the frictional forces will also differ, and the slider moves farther in one direction than in the other one.

Let us look how this phenomenon can be described analytically.

2.4.4 Transformation to the Principal Form. Amplitude of the Resonator

In this section we are going to split the system (2.97) into two parts. The first one will describe the oscillations of the resonator and the second one will govern the transportation of the slider. The first step is to solve (2.97) with respect to the highest derivatives. The result is:

$$
\begin{aligned}
&w'' + \frac{\left(2\beta w' + w\right)}{1 - \dfrac{\alpha\mu_\Delta \sin\theta\cos\theta}{\left(1-\alpha\cos^2\theta\right)}} \\
&\quad = a\Omega^2 \sin\Omega t + \frac{\left(\varepsilon_f\left(p+\sin\psi\right) + \bar{\mu}\alpha n_0 + \mu_\Delta \varepsilon_f \cos\psi\right)\cos\theta}{1 - \dfrac{\alpha\mu_\Delta \sin\theta\cos\theta}{\left(1-\alpha\cos^2\theta\right)}} \\
&x'' = \frac{\alpha\left(2\beta w' + w\right)\left(\cos\theta + \mu_\Delta \sin\theta\right)}{1 - \dfrac{\alpha\mu_\Delta \sin\theta\cos\theta}{\left(1-\alpha\cos^2\theta\right)}} \\
&\qquad - \frac{\varepsilon_f\left(p+\sin\psi\right) + \bar{\mu}\alpha n_0 + \mu_\Delta \varepsilon_f \cos\psi}{1 - \dfrac{\alpha\mu_\Delta \sin\theta\cos\theta}{\left(1-\alpha\cos^2\theta\right)}}
\end{aligned}
\tag{2.99}
$$

Next, we do assumptions about the magnitude orders of the parameters. Introducing a formal small parameter $\varepsilon << 1$ we assume that

$$
\alpha = \varepsilon\alpha_0; \bar{\bar{\mu}} = \varepsilon\bar{\bar{\mu}}_0; \tilde{\bar{\mu}} = \varepsilon\tilde{\bar{\mu}}_0; a\Omega^2 = \varepsilon^2 a_0; \beta = \varepsilon^2\beta_0; \varepsilon_f = \varepsilon^3\varepsilon_{f0} \tag{2.100}
$$

Here all the parameters $\alpha_0, \bar{\bar{\mu}}_0, \tilde{\bar{\mu}}_0, a_0, \beta_0, \varepsilon_{f0}$ have the magnitude order of one. By this, we quantify the mass of the resonator, the friction, the amplitude of excitation, the damping, and the inverse natural frequency of the resonator in their magnitude orders.

Keeping terms to $O\left(\varepsilon^3\right)$ we can rewrite (2.99) as follows:

$$
\begin{aligned}
&w'' + 2\varepsilon^2\beta_0 w' + w + \varepsilon^2\alpha_0\bar{\bar{\mu}}\,\mathrm{sgn}\left(x'\right)qw = \\
&\quad \varepsilon^2 a_0 \sin\Omega\tau + \left(\varepsilon^3\varepsilon_{f0}\left(p+\sin\psi\right) + \varepsilon^2\alpha_0\bar{\bar{\mu}}\,\mathrm{sgn}\left(x'\right)n_0\right)\cos\theta \\
&x'' = \varepsilon\alpha_0 w\cos\theta + \varepsilon^2\alpha_0\tilde{\bar{\mu}}\,\mathrm{sgn}\left(x'\right)w\sin\theta \\
&\qquad\qquad - \left(\varepsilon^3\varepsilon_{f0}\left(p+\sin\psi\right) + \varepsilon^2\alpha_0\bar{\bar{\mu}}\,\mathrm{sgn}\left(x'\right)n_0\right) \\
&q \equiv \frac{\sin\theta\cos\theta}{1-\alpha\cos^2\theta}
\end{aligned}
\tag{2.101}
$$

Consider the unperturbed system to (2.101). More precise formulated we neglect all the terms smaller than $O\left(\varepsilon\right)$:

$$\begin{aligned} w_0'' + w_0 &= 0 \\ x_0'' &= \varepsilon\alpha_0 w_0 \cos\theta \end{aligned} \tag{2.102}$$

Its general solution can be taken as a variable's transformation for the complete system (2.101):

$$\begin{aligned} w &= u\sin\varphi, \quad w' = u\cos\varphi \\ x' &= \varepsilon^2\alpha_0 v - \varepsilon\alpha_0 u\cos\varphi\cos\theta \end{aligned} \tag{2.103}$$

The idea of this transformation is simple: The first equation in (2.101) is quasi-linear; so the standard Van der Pol transformation introduces the amplitude u of the resonator's vibrations, having rate of change $O\left(\varepsilon^2\right)$ (i.e. it becomes a slowly varying variable), and the phase φ, which becomes a fast variable. Similarly, the transformation from x' to v serves to eliminate the dominating first term on the right-hand side of the second equation in (2.101). Thus v expresses a small time varying correction to the first order solution to the unperturbed problem, i.e. the approximate solution obtained when neglecting terms of $O\left(\varepsilon^2\right)$ and higher.

It is also important to notice that in this approximation the motion of the resonator is independent from the motion of the slider. Thus the system is split into two weakly interacting subsystems.

In the new variables the functions expressing friction become:

$$\begin{aligned} \bar{\mu}\left(x'\right) &= \varepsilon\bar{\bar{\mu}}_0 \cdot \operatorname{sgn}\left(\varepsilon v - u\cos\varphi\cos\theta\right) \\ \mu_\Delta\left(x'\right) &= \varepsilon\bar{\tilde{\mu}}_0 \cdot \operatorname{sgn}\left(\varepsilon v - u\cos\varphi\cos\theta\right) \end{aligned} \tag{2.104}$$

We are going to investigate the resonant case, so we can assume $\delta = 1 - \Omega$ to be small, i.e. $\delta = \varepsilon^2\delta_0$, where $\delta_0 = O(1)$. Further we introduce a new variable $\sigma = \varphi - \Omega t$ that expresses the phase difference between the excitation and the oscillations of the resonator. (This transformation is typical for almost linear resonance problems [23].) Then the system (2.101) takes the following form (we keep the terms up to $O\left(\varepsilon^2\right)$):

$$
\begin{aligned}
u' &= -2\varepsilon^2\beta_0 u\cos^2\varphi + \varepsilon^2 a_0 \sin(\varphi-\sigma)\cos\varphi \\
&\quad -\varepsilon^2\alpha_0\left(\bar{\bar{\mu}}qu\sin\varphi - \bar{\bar{\mu}}n_0\cos\theta\right)\operatorname{sgn}\left(\varepsilon v - u\cos\varphi\cos\theta\right)\cos\varphi \\
\sigma' &= \varepsilon^2\delta_0 + 2\varepsilon^2\beta_0\sin\varphi\cos\varphi - \frac{\varepsilon^2\alpha_0}{u}\sin(\varphi-\sigma)\sin\varphi \\
&\quad + \frac{\varepsilon^2\alpha_0}{u}\left(\bar{\bar{\mu}}qu\sin\varphi - \bar{\bar{\mu}}n_0\cos\theta\right)\operatorname{sgn}\left(\varepsilon v - u\cos\varphi\cos\theta\right)\sin\varphi \\
\varphi' &= 1 + 2\varepsilon^2\beta_0\sin\varphi\cos\varphi - \frac{\varepsilon^2\alpha_0}{u}\sin(\varphi-\sigma)\sin\varphi \\
&\quad + \frac{\varepsilon^2\alpha_0}{u}\left(\bar{\bar{\mu}}qu\sin\varphi - \bar{\bar{\mu}}n_0\cos\theta\right)\operatorname{sgn}\left(\varepsilon v - u\cos\varphi\cos\theta\right)\sin\varphi \\
v' &= \left(\bar{\bar{\mu}}u\sin\varphi\sin\theta - \bar{\bar{\mu}}n_0\right)\operatorname{sgn}\left(\varepsilon v - u\cos\varphi\cos\theta\right) \\
&\quad -\varepsilon\varepsilon_{f0}\left(p+\sin\psi\right) + \varepsilon^2 R(u,\sigma,\varphi,v)
\end{aligned}
\tag{2.105}
$$

Here the function $\varepsilon^2 R$ contains small terms having zero time average (these terms are unimportant for the following considerations).

The system (2.105) is ready for asymptotic analysis. There are two slow variables: u and σ, expressing the amplitude of resonator oscillations and the phase difference between these oscillations and the excitation; they are slowly changing because u' and σ' are $O\left(\varepsilon^2\right)$. There is also a fast rotating phase $\varphi : \varphi' = O(1)$.

Variable v is the most interesting one for the present study, because it describes the velocity of the slider. This variable differs in character from the other ones. Firstly, it is not a slow variable, since the first term in the equation for v' is not necessarily small. Secondly, this dominating term is discontinuous due to the "sign"-functions. Nevertheless, v does not influence the motion of the resonator significantly, because it occurs only in the equations in the form of a small combination εv. So we can find the resonator's amplitude simply by averaging the first two equations of (2.105) with respect to the fast rotating phase φ.

In the averaging process we need the following mean values:

$$\begin{aligned}
&\left\langle \operatorname{sgn}\left(\cos\varphi\right)\sin\varphi\cos\varphi\right\rangle = 0;\ \left\langle \operatorname{sgn}\left(\cos\varphi\right)\sin\varphi\right\rangle = 0\\
&\left\langle \operatorname{sgn}\left(\cos\varphi\right)\sin\varphi\cos^2\varphi\right\rangle = 0;\ \left\langle \operatorname{sgn}\left(\cos\varphi\right)\cos 2\varphi\right\rangle = 0\\
&\left\langle \operatorname{sgn}\left(\cos\varphi\right)\sin^2\varphi\right\rangle = 0;\left\langle \operatorname{sgn}\left(\cos\varphi\right)\cos^2\varphi\right\rangle = 0\\
&\left\langle \operatorname{sgn}\left(\cos\varphi\right)\cos\varphi\right\rangle = \frac{2}{\pi}
\end{aligned} \tag{2.106}$$

Hence, we obtain the averaged equations for the resonator, with subscript 1 denoting averaged variables:

$$\begin{aligned}
u_1' &= -\varepsilon^2\beta_0 u_1 - \frac{1}{2}\varepsilon^2 a_0\sin\sigma_1 - \frac{2\varepsilon^2\alpha_0\overline{\overline{\mu}}_0}{\pi}n_0\cos\theta\\
\sigma_1' &= \varepsilon^2\delta_0 - \varepsilon^2\frac{a_0}{2u_1}\cos\sigma_1
\end{aligned} \tag{2.107}$$

If we are interested only in the stationary solutions of (2.107), we must put its right-hand sides to zero. Solving these equations, we obtain the stationary amplitude of the resonator's oscillations:

$$\begin{aligned}
u_{1_St} = &-\frac{2\alpha_0\overline{\overline{\mu}}_0\beta_0 n_0\cos\theta}{\pi\left(\beta_0^2+\delta_0{}^2\right)}\\
&+\sqrt{\frac{a_0^2}{4\left(\beta_0^2+\delta_0{}^2\right)}-\left(\frac{2\alpha_0\overline{\overline{\mu}}_0\beta_0 n_0\cos\theta}{\pi\left(\beta_0^2+\delta_0{}^2\right)}\right)^2}
\end{aligned} \tag{2.108}$$

These equations describe the oscillations of the resonator to the first order, but they still do not give us any information about motions of the slider.

Next we attempt to find an appropriate approximation for the slider's velocity.

2.4.5 Motion of the Slider: Preparing for Averaging

The slider's velocity v is governed by the last equation in (2.105):

$$\begin{aligned}
v' = &\left(\tilde{\overline{\mu}}_0 u\sin\theta\sin\varphi - \overline{\overline{\mu}}_0 n_0\right)\operatorname{sgn}\left(\frac{\varepsilon v}{u\cos\theta}-\cos\varphi\right)\\
&-\varepsilon\varepsilon_{f0}\left(p+\sin\psi\right)+\varepsilon^2 R
\end{aligned} \tag{2.109}$$

We consider (2.109) as the perturbed equation for an unperturbed equation obtained by omitting terms of $O(\varepsilon)$ and smaller:

$$v_0' = \left(\bar{\bar{\mu}}_0 n_0 - \bar{\tilde{\mu}}_0 u \sin\theta \sin\varphi\right) \operatorname{sgn}\left(\cos\varphi\right) \tag{2.110}$$

This equation can be solved quite easily. Let us introduce two functions (*cf.* Fig. 2.25):

$$\begin{aligned} \Pi(\varphi) &= \int_0^{\varphi} \operatorname{sgn}\left(\cos\psi\right) d\psi = \arcsin\left(\sin\varphi\right) \\ 1 - \left|\cos\varphi\right| &= \int_0^{\varphi} \sin\psi \, \operatorname{sgn}\left(\cos\psi\right) d\psi \end{aligned} \tag{2.111}$$

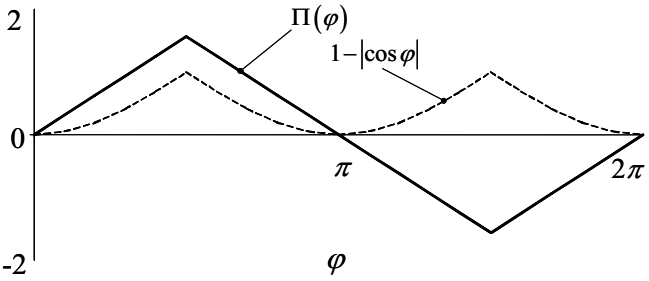

Fig. 2.25. Functions $\Pi(\varphi)$ and $1-\left|\cos\varphi\right|$

Using these functions we can find the solution to (2.110) in the following form:

$$v_0 = n_0 \bar{\bar{\mu}}_0 \Pi(\varphi) + \bar{\tilde{\mu}}_0 u \sin\theta \left(\left|\cos\varphi\right| - \frac{2}{\pi} \right) \tag{2.112}$$

This solution can be used as the basis for the transformation of the perturbed equation. We try to find the slider's velocity as a sum of v_0 and a small correction:

$$v = v_0 + v_p \tag{2.113}$$

Inserting (2.113) into (2.109), one finds that the new variable v_p is governed by the following equation:

$$v_p' = \left(\bar{\bar{\mu}}_0 n_0 - \bar{\tilde{\mu}}_0 u \sin\theta \sin\varphi\right) \times \left\{\operatorname{sgn}\left[\cos\varphi - \varepsilon\Phi\left(u,\varphi,v_p\right)\right] - \operatorname{sgn}(\cos\varphi)\right\} - \varepsilon\varepsilon_{f0}\left(p + \sin\psi\right) + O\left(\varepsilon^2\right) \tag{2.114}$$

$$\Phi\left(u,\varphi,v_p\right) = \frac{\bar{\bar{\mu}}_0 n_0 \Pi(\varphi) + \bar{\tilde{\mu}}_0 u \sin\theta\left(|\cos\varphi| - \frac{2}{\pi}\right) + v_p}{u\cos\theta}$$

This equation is very interesting and worth discussing in detail. Its right hand side is not small. The first term is proportional to the difference between two sign-functions. It is important that the arguments of these functions differ only slightly. Such a situation is illustrated in Fig. 2.26.

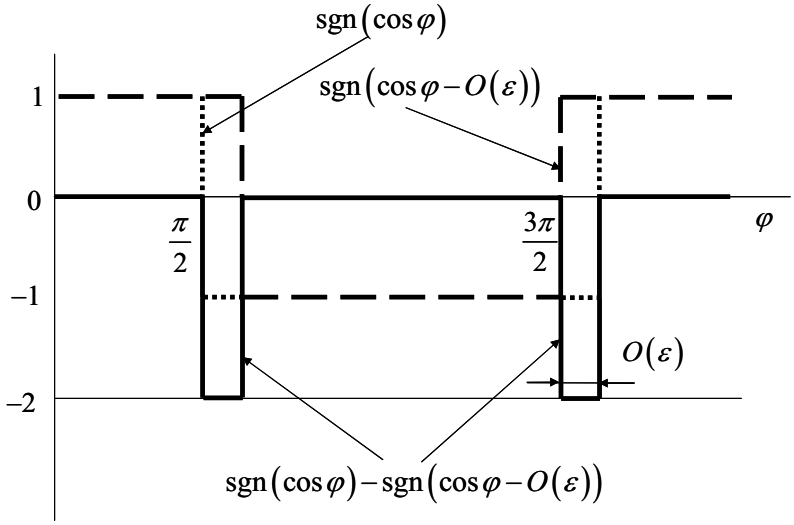

Fig. 2.26. Difference between two sign-functions

The difference between these functions is almost always equal to zero. Only in two short time intervals it is equal to -2 (it can be also equal to 2, depending on the sign of the small arguments' deviation). But in an integral sense, the effect of such a term is as small as the effect of any continuous small term of the same order of magnitude as the difference between the arguments of the sign-functions. Notice that the zero of $\cos\varphi$ is a simple one (cf. the discussion of the Theorem 1). So it seems plausible that the equation (2.114) can be averaged with the similar accuracy as an equation in which the right hand side is multiplied by the small parameter. This idea can be formulated as a theorem.

Theorem 2.2

Consider the initial value problem:

$$\begin{aligned} &\dot{x} = \varepsilon X\left(x,t\right) + Z\left(x,t\right)\left[E\left(g\left(t\right) + \varepsilon f\left(x,t\right)\right) - E\left(g\left(t\right)\right)\right] \\ &x\left(0\right) = x_0 \end{aligned} \tag{2.115}$$

Here the functions $X, Z : R^{n+1} \to R^n; f : R^{n+1} \to R^1; g : R^1 \to R^1$ *are defined for* $x, \xi, x_0 \in D \subset R^n, t \in \left[0, \infty\right), \varepsilon \in \left(0, \varepsilon_0\right]$. $E\left(t\right)$ *is a one-step function.*

Consider the averaged problem alongside (2.115):

$$\begin{aligned} &\dot{\xi} = \varepsilon \Xi\left(\xi\right) + \varepsilon \varsigma\left(\xi\right) \\ &\xi\left(0\right) = x_0 \\ &\Xi\left(\xi\right) = \left\langle X\left(\xi,t\right)\right\rangle \\ &\varepsilon \varsigma\left(\xi\right) = \left\langle Z\left(\xi,t\right)\left[E\left(g\left(t\right) + \varepsilon f\left(\xi,t\right)\right) - E\left(g\left(t\right)\right)\right]\right\rangle \end{aligned} \tag{2.116}$$

Suppose

1. X, Z *and* f *are measurable functions of* t *for constant* x *and* ε.

2. f *in addition is a piecewise differentiable function with respect to* t, *and its derivative is bounded*

3. All the functions are T*-periodic with respect to* t. $\left\langle \ \right\rangle$ *means the time average.*

4. X, Z *and* f *are bounded Lipschitz-continuous functions in* x *on* D.

5. $g\left(t\right) \in C^1$, *i.e. it is a differentiable function.*

6. An equation $g\left(t\right) = 0$ *has* m *simple solutions* t_{i0}, $i = 1, \ldots, m$ *for* $t \in \left[0, T\right)$.

7. The first derivative of the function g *in these points is not equal to zero:*

$$\left|g'\left(t_{i0}\right)\right| \geq G > 0 \tag{2.117}$$

8. All constants do not depend on ε, *and* ξ *belongs to the interior subset of* D *on the time scale* $1/\varepsilon$.

Then the solutions of (2.115) and (2.116) are asymptotically close to each other, i.e. the error one makes on using the averaged system instead of the original one is small for the asymptotically long time interval:

$$\left\| x(t) - \xi(t) \right\| \le c_3 \varepsilon e^{c_4 \varepsilon t} \tag{2.118}$$

The proof of this theorem can be found in Appendix III.

Let us apply this theorem to the equation (2.114) describing the motion of the slider. The only thing we have to do is simply average (2.114). As a result we obtain the following approximate equation for the slider, with v_1 denoting averaged values:

$$v_1' = -\frac{2\varepsilon n_0}{\pi u_1 \cos\theta} v_1 + \left(1 + \frac{4}{\pi^2}\right) \varepsilon \bar{\bar{\mu}}_0 n_0 \tan\theta - \varepsilon \frac{\varepsilon_{f0}}{\bar{\bar{\mu}}_0} \left(p + \sin\psi\right) \tag{2.119}$$

The stationary value of the slider's average velocity can be found by letting the right-hand side of (2.119) to zero, giving the stationary value for v_1 – and then inserting (2.103) in order to obtain the corresponding value for the slider's velocity x'. The result is as follows:

$$v_{St} = \frac{\pi \alpha_0 u_{1\_St} \cos\theta}{2 n_0} \left[\left(1 + \frac{4}{\pi^2}\right) \bar{\bar{\mu}}_0 n_0 \tan\theta - \frac{\varepsilon_{f0}}{\bar{\bar{\mu}}_0} \left(p + \sin\psi\right) \right] \tag{2.120}$$

Here u_{1_St} is the stationary value for the resonator's amplitude given in (2.108).

We have now obtained a simple expression for the average (non-sticking) motions of the slider. In the next section we compare this prediction to numerical results and discuss the influence of a system's parameters.

2.4.6 Performance in Dependence of Parameters; Comparison between Analytic Prediction and Numerical Simulations

Fig. 2.27. (a) – (f) shows the variation of the average slider velocity v_{St} with some essential system parameters. The curves are based on the approximate expression (2.120) with default values of the non-dimensional parameters as given in the figure legend; for each curve these values were taken, while one of the parameters was varied. Circle markers show results obtained by numerical integration of the full equations of motion (2.97). The results between the approximate and numerical methods agree very well. This justifies the rather complicated but correct way to obtain the averaged solution in comparison with a simpler but not so accurate approach represented in [122].

Fig. 2.27(a) shows a maximum for the averaged slider's velocity, when the excitation frequency Ω is close to the natural frequency of the internal resonator. This curve reflects the resonant character of the investigated phenomenon.

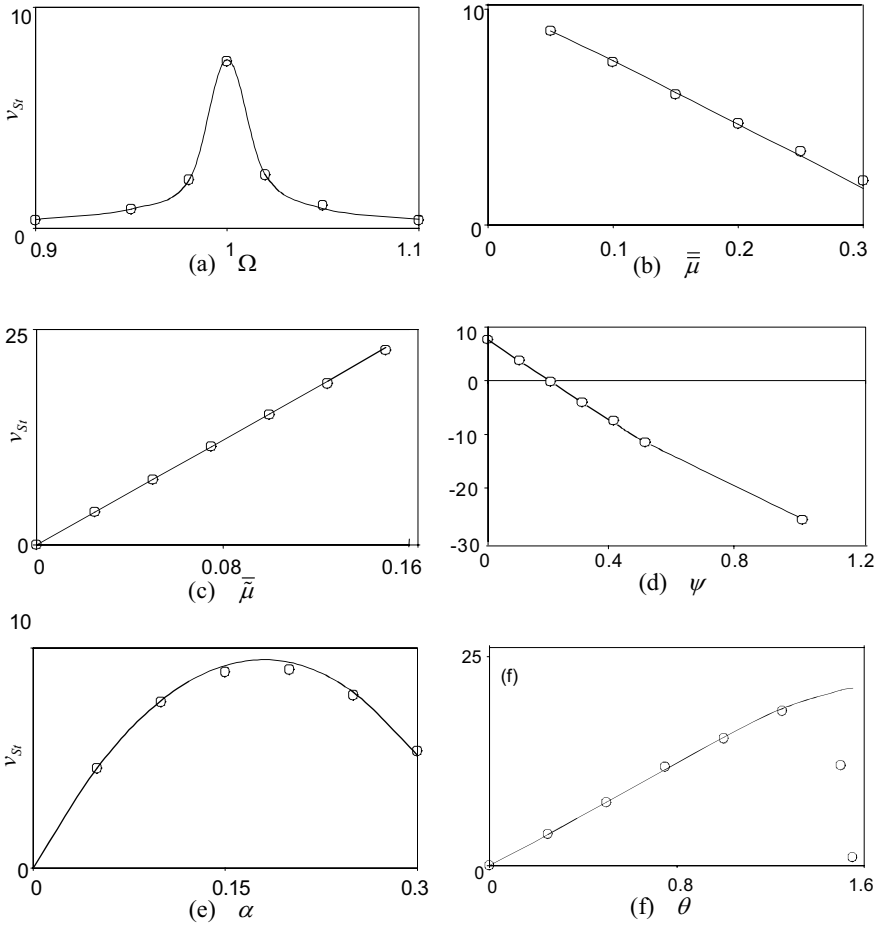

Fig. 2.27. Variation of the average slider's velocity v_{St} with parameters; straight line corresponds to analytic prediction, circles show the results of the numerical simulations; default parameters values: $\varepsilon = 0.1$, $\alpha_0 = 1$, $\bar{\bar{\mu}}_0 = 1$, $\tilde{\bar{\mu}}_0 = 0.25$, $\varepsilon_{f\,0} = 1$, $\beta_0 = 0.5$, $a_0 = 2$; $p = 0$, $\psi = 0$, $n_0 = 0.5$, $\theta = 0.5$

Fig.2.27(b) displays the effect of the average friction coefficient. It was varied across a wide range in order to show the main tendency: The average velocity of the slider decreases almost linearly with increased average friction.

Fig.2.27(c) shows an even stronger influence of the asymmetry in friction, i.e. the difference between friction coefficients of the two friction layers surrounding

the slider. This asymmetry is the essential source of the whole effect causing the slider to move, so if $\bar{\tilde{\mu}} = 0$ the slider's velocity vanishes.

Fig.2.27(d) shows the ability of the slider to climb against gravity. For the actual parameters, it appears, this is possible for stator inclinations up to about 11° (≈ 0.2 rad), at which the *vibration induced forces* just balance gravity and the slider reverses its (average) direction of motion from upwards to downwards.

Fig.2.27(e) displays the influence of the resonator mass ratio α, which is not trivial. The curve has a characteristic maximum, which can be explained qualitatively: Slider motions are caused by forces induced by the resonator. Therefore, it seems obvious that increasing the mass of the resonator will increase the velocity of the slider. However, an increase in resonator mass may also decrease the resonators oscillation amplitude. At some level of α this effect overcomes the first one, and the slider's velocity starts to decrease.

Finally, Fig.2.27(f) depicts how v_{St} is affected by the resonator inclination θ. If $\theta = 0$ the asymmetry in friction has no effect on the excitation, so the slider has zero average velocity. With increasing θ the asymmetry in friction comes into play, and v_{St} increases almost linearly up to about 75° (≈ 1.3 rad) in the figure. Beyond this value, the friction forces become large enough to cause periodical sticking of the slider, which lowers the average velocity as appears from the numerical results (circles). The approximate approach used here does not take into account the possibility of sticking. Therefore, the approximate expression (2.120) consequently predicts erroneously large slider velocities as $\theta \to 90^\circ$ (≈ 1.57 rad). Consideration of sticking motions might be included by applying the variable order averaging (*cf.* Chapter 4).

2.4.7 Conclusions

The example investigated is interesting both from practical and theoretical point of view.

Firstly, it has allowed us to demonstrate how the higher order approximations can be obtained for the constant order regimes in discontinuous systems. It was demonstrated that averaging can be used not only in systems with small right hand sides, but it is also applicable for systems in which right hand sides are large but only during short time intervals. (Really important is, how small the corresponding integral is.)

Secondly, the investigated slider can be used as a moving transport device in many difficult situations like pipes or in medical applications. Its advantage is the simple excitation mechanism, the ability to climb against gravity and the ultrasonic frequency of oscillations with extremely small amplitude. Due to the last fact, the oscillations (which can be neither seen nor heard) do not disturb even in very delicate situations.

3. Systems with Almost Elastic Collisions

Systems with collisions differ significantly from the considered systems with friction. First of all, systems with collisions cannot be described by ordinary differential equations at all times. Ordinary differential equations are valid between the collisions, but some additional kinematic conditions must be used in order to calculate state variables after each collision, as functions of the state variables before the collision. It is typical that the coordinates of the colliding objects remain continuous in oscillating systems with collisions. The velocities on the contrary become discontinuous. The velocities change during a collision is usually not small – it has the same magnitude order as the velocities themselves. That is the main reason why general perturbation methods, like multiple scales or classical averaging developed for ordinary differential equations, cannot be applied directly to analyze systems with collisions.

It should be distinguished between periodic, transient and chaotic motions in systems with collisions. The recent interest to chaotic motions in discontinuous and impact oscillators [3, 13, 32, 41, 46 – 48, 50, 57, 88 – 92, 100 – 104, 106, 107, 115, 116, 134] is mainly motivated by the unwanted vibrations, like gear rattling [28, 57, 92, 93] or liquid sloshing [94]. For machines using vibrations like vibration crushers, screens or grinding machines the simplest periodic regimes are of the main interest.

Different mathematical methods are used for analyzing these simplest regimes. The classical approach was developed for the piecewise linear systems. According to this approach the equations of motions have to be integrated between the collisions, and the kinematic conditions describing collisions are used to stitch one interval of the solution to another one. In simple cases without additional nonlinearities this approach provides deep understanding for the variety of dynamic behavior in systems with collisions [1, 58]. It is also very effective in numerical simulations. However “stitching” becomes very elaborate for obtaining analytical predictions in presence of additional nonlinearities.

Another method useful for numeric realization is mapping [44]. This method was also successfully used in order to investigate the existence and stability of periodic solutions. For example the completely nonlinear problem of self-synchronization in systems with impacts was considered using this method [136].

Further development of analytic methods was connected with the ideas of a small parameter and perturbations. Poincare`s method, historically the first method of small parameter, is very effective in order to analyze periodic solutions. It was generalized for the discontinuous systems of constant order by Kolovsky [59, 60] and for discontinuous systems of variable order by Nagaev [77]. These

methods helped to analyze existence and stability of periodic solutions in slightly nonlinear systems with impacts, but the necessity to stabilize the technologically best impact regimes made the transient analysis inevitable.

Very tempting was the idea to apply harmonic linearization for the transient analysis of impact oscillators [9, 10]. This approach is very easy but not validated. Thus its results are somehow unreliable. Numerous comparisons of the corresponding approximate results for the simplest piecewise linear systems with exact and numeric solutions [10] shows that harmonic linearization is accurate enough for systems with strong filtering properties of elastic structures. The discrepancy increases with increasing energy dissipation and outside of the very small vicinity of resonance. Another point, especially noticeable in [9], is that the harmonic linearization does not distinguish between different dynamical regimes, i.e. the variety of impact oscillations vanishes completely.

One of the effective approaches, to apply averaging to systems with collisions is discussed in this chapter. The basic idea is to transform the corresponding equations of motion, including kinematic relationships describing collisions, to a form for which the modified averaging procedure can be established and validated. The main idea is to extrude the discontinuity from the equations applying a transformation containing the essential discontinuity. This idea was suggested in [137 – 139] and developed for different applications in [33, 52]. The original objective was to eliminate the discontinuity completely. Then it would be possible to apply standard averaging. This idea being quite successful for perfectly elastic collisions leads to artificial transformations with unclear physical sense even in case of a small energy dissipation during collisions. On the other hand it is not really necessary to eliminate the discontinuities completely. It is enough to reduce them to a sufficiently small level relative to other damping sources and to generalize the averaging technique for this case. This combination leads to an efficient approach for asymptotic analysis of colliding oscillators.

Two different types of oscillating systems with almost elastic collisions are considered in this chapter:

– systems limited at one side (a mass near a wall)
– systems limited at both sides (a mass in a clearance)

The basic idea for the unfolding transformations in these two cases is discussed in paragraph 3.1. The classical “mass on moving belt” with one side limit is considered in paragraphs 3.2 and 3.3 as an example for the first and the second order averaging for a system with small discontinuity. The same system in a clearance is analyzed in paragraph 3.4. The classical resonantly excited impact oscillator limited at one side is considered in paragraph 3.5. The same system in a clearance is investigated in paragraph 3.6. Mathematical basics for the described averaging procedure are given in Appendix 5 which can be omitted by readers interested primarily in mechanical and technical applications but is strongly recommended to those interested in further development of the discussed methods.

3.1 The Basic Ideas of Discontinuous Averaging. Unfolding Transformations

It is not possible to apply standard averaging or other perturbation methods in order to obtain approximate solutions to systems with collisions, because these methods require the right hand sides of the equations of motion to be sufficiently smooth. The basic idea of the discontinuous system's analysis is to find another kind of "standard form" to which discontinuous systems can be transformed and for which some kind of averaging can be validated. The most straightforward way to do this is to apply the discontinuous transformation technique.

3.1.1 The Basic Idea of the Unfolding Transformation for the Mass Limited at One Side

The main idea becomes quite clear if one looks at a typical solution for an elastic impact oscillator without external forces (see Fig. 3.1).

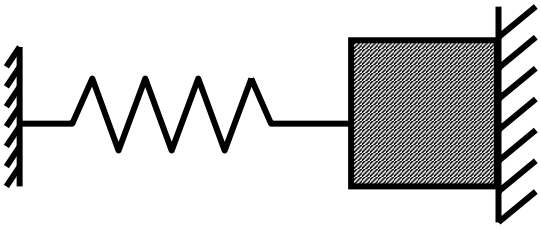

Fig. 3.1. An oscillator with an elastic limit at one side

Its motion is governed by equations:

$$\begin{aligned} &\ddot{x} + x = 0,\ x < 0 \\ &x = 0 \quad \Rightarrow \qquad x_+ = x_-;\ \dot{x}_+ = -\dot{x}_- \end{aligned} \tag{3.1}$$

The sense of these equations is quite simple. It describes a system containing of a mass and a spring. The motion of the mass is limited by an absolutely elastic wall placed in the equilibrium point. The mass colliding against the wall reflects without any loss of kinetic energy (the velocity magnitude remains constant and its direction changes).

The general solution to this system is represented in Fig. 3.2.

$$x = -A\left|\sin\left(t + \varphi\right)\right| \tag{3.2}$$

This solution is nothing different than a folded sin-function (see Fig. 3.3). So the system (3.1) can be obviously regularized by a non-smooth transformation:

$$x = -|z| \qquad \Rightarrow \qquad \ddot{z} + z = 0 \tag{3.3}$$

This transformation automatically fulfils the elastic impact's conditions and transforms (unfolds) the nonlinear system (3.1) to an elementary linear oscillator. The same idea for eliminating impacts by means of a non-smooth transformation already containing the essential discontinuity, was generalized in [52] for the case of almost elastic collisions ($1 - R << 1$). But actually it is not necessary to eliminate the impact discontinuity completely. If we are going to apply a generalized averaging method, it is absolutely sufficient to reduce the variable jumps to the same magnitude order as other small terms.

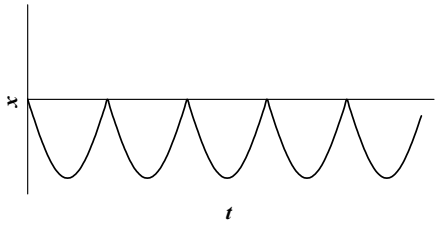

Fig. 3.2. A typical solution to (3.1) is a folded sinus-function

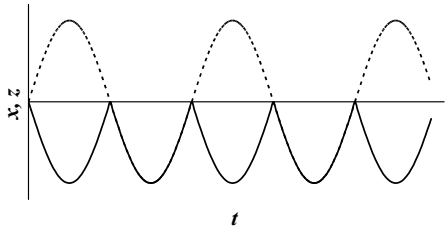

Fig. 3.3. A solution to (3.1) in the folded and unfolded forms

3.1.2 The Unfolding Transformation and Averaging in Case of Slightly Inelastic Collisions

This idea can be illustrated by an elementary generalization to (3.1), taking both linear damping between the collisions and the imperfect velocity restitution in the collisions, into account. The system differs from (3.1) because the limiter is no longer located in the equilibrium point. Instead it is shifted slightly away from this position. The distance between the equilibrium position of the mass (without limiter) and the wall is Δ. Consider the following system:

$$\begin{aligned} &\ddot{x}+\beta\dot{x}+x=0, \quad x<\Delta \\ &x=\Delta \quad \Rightarrow \qquad x_{+}=x_{-}; \quad \dot{x}_{+}=-R\dot{x}_{-}; \quad 0<R<1 \end{aligned} \tag{3.4}$$

The same unfolding transformation (3.3) can be applied here with an insignificant modification.

$$x=\Delta-|z|, \; z_{-}\cdot z_{+}<0 \tag{3.5}$$

This transformation expresses the physical condition of the mass not penetrating the limiter.

The inequality guarantees the unfolding character and the uniqueness of the transformation. This obvious inequality is assumed in all following transformations although it is not written down explicitly.

It is easy to obtain the following equations for the unfolded variable:

$$\begin{aligned} &\ddot{z}+\beta\dot{z}+z=\Delta\,\mathrm{sgn}\,z, \; z\neq 0 \\ &z=0 \quad \Rightarrow \qquad z_{+}=z_{-}; \; \dot{z}_{+}=R\dot{z}_{-} \end{aligned} \tag{3.6}$$

The main difference between equations (3.4) and (3.6) is that the change of the unknown function during each collision in (3.6) is small for almost elastic collisions: $(1-R)<<1$. This statement becomes obvious if we rewrite the kinematic relationships describing collision in (3.6) as follows:

$$z=0 \quad \Rightarrow \qquad z_{+}-z_{-}=0; \; \dot{z}_{+}-\dot{z}_{-}=-(1-R)\dot{z}_{-} \tag{3.7}$$

Now the assumptions about the magnitude order of the parameters have to be taken. It is quite natural for perturbation methods to assume that the dissipative parameters and the shift between the equilibrium point of the spring and the limiter are small parameters of the same magnitude order.

$$\beta<<1; \; (1-R)<<1; \; \Delta<<1 \tag{3.8}$$

The unperturbed system corresponding to (3.6) is a linear harmonic oscillator under these assumptions. The standard Van-der-Pol transformation can be used in order to introduce slowly varying amplitude and a fast "rotating" phase. (Notice that the amplitude is assumed to be always positive.)

$$\begin{aligned} &z=A\sin\varphi; \; \dot{z}=A\cos\varphi \\ &\dot{A}=-\beta A\cos^{2}\varphi+\Delta\cos\varphi\,\mathrm{sgn}(\sin\varphi), \quad \varphi\neq\pi n \\ &A_{+}-A_{-}=-(1-R)A_{-}, \; \varphi=\pi n \\ &\dot{\varphi}=1+\beta\sin\varphi\cos\varphi-\frac{\Delta}{A}|\sin\varphi| \end{aligned} \tag{3.9}$$

Taking A as the new unknown function and φ as the new independent variable the following system can be obtained:

$$\begin{cases} \dfrac{dA}{d\varphi} = \dfrac{-\beta A\cos^2\varphi + \Delta\cos\varphi\,\mathrm{sgn}\left(\sin\varphi\right)}{1+\beta\sin\varphi\cos\varphi - \dfrac{\Delta}{A}\left|\sin\varphi\right|}, \quad \varphi \neq \pi n \\ A_+ - A_- = -\left(1-R\right)A_-, \quad \varphi = \pi n \end{cases} \tag{3.10}$$

The unknown function A here is slow (its derivative has the magnitude order of the small parameter) between collisions and its change while colliding is also small.

An averaging procedure can be validated for systems in this form. The basic theorem for the first order averaging in this case was formulated in [139].

Theorem 3.1.
Consider a system

$$\begin{aligned} &\frac{dx}{d\varphi} = \varepsilon X\left(\varphi, x\right) \text{ if } \varphi \neq \pi n \\ &\Delta x \equiv x_+ - x_- = \varepsilon f\left(x_-\right) \text{ if } \varphi = \pi n \end{aligned} \tag{3.11}$$

Here $x \in D \subset R^n$, x_- *and* x_+ *are the values of the phase variables before and after the passage of the independent variable* φ *through the value* πn.
Suppose

1. X *is a measurable function of* φ *for constant* x *and* ε.
2. *Function* X *is* 2π *-periodic with respect to* φ. $\left\langle\ \right\rangle_\varphi$ *means the average with respect to the fast rotating phase* φ*:*

$$\left\langle g\left(\varphi\right)\right\rangle_\varphi = \frac{1}{2\pi}\int_0^{2\pi} g\left(\varphi\right)d\varphi \tag{3.12}$$

3. X *is a bounded Lipschitz-continuous function in* x *on* D, *i.e.*

$$\begin{aligned} &\left\|X(x,\varphi,\varepsilon)\right\| \le M_X; \\ &\left\|X(x_1,\varphi,\varepsilon) - X(x_2,\varphi,\varepsilon)\right\| \le \lambda_X\left\|x_1 - x_2\right\| \end{aligned} \tag{3.13}$$

4. $f(x)$ *is bounded and Lipschitz-continuous on* D.

5. *All constants do not depend on* ε.

6. *The solution* ξ *of the averaged system belongs to the interior subset of* D *on the* φ *scale* $1/\varepsilon$.

$$\frac{d\xi}{d\varphi} = \varepsilon\Xi(\xi) + \frac{\varepsilon}{\pi} f(\xi),\ \ \Xi(\xi) = \langle X(\varphi,\xi)\rangle_{\varphi} \tag{3.14}$$

Under these conditions solutions of the initial value problems (3.11) *and* (3.14) *are asymptotically close to each other, i.e.*

$$\|x(\varphi) - \xi(\varphi)\| \le \varepsilon O(1), \quad for\ \varphi = O(1/\varepsilon) \tag{3.15}$$

The proof of this theorem can be found in Appendix V. Here we are going to use it in order to obtain an approximate solution to our problem. If one is interested in the first order approximation only, the smaller terms can be neglected and the equations (3.10) rewritten as follows:

$$\begin{cases} \dfrac{dA}{d\varphi} = -\beta A\cos^2\varphi + \Delta\cos\varphi\,\mathrm{sgn}(\sin\varphi), & \varphi \ne \pi n \\ A_+ - A_- = -(1-R)A_-, & \varphi = \pi n \end{cases} \tag{3.16}$$

Averaging this system according to (3.14) the following first order approximation can be easily obtained:

$$\frac{dA_1}{d\varphi} = -\left(\frac{\beta}{2} + \frac{1-R}{\pi}\right)A_1. \tag{3.17}$$

Index one indicates the first order approximation. This is a quite simple linear equation. Its general solution is

$$A_1 = Ce^{-\beta_{eff}\varphi};\ \beta_{eff} = \frac{\beta}{2} + \frac{1-R}{\pi} \tag{3.18}$$

The arbitrary constant C is determined by the initial conditions. The same averaging procedure can be applied to the system (3.9) directly. Then it is possible to obtain the first order approximations both for the amplitude and the phase.

$$\dot{A}_1 = -\beta_{eff}A_1;\ \ \dot{\varphi}_1 = 1 - \frac{2\Delta}{\pi A_1}. \tag{3.19}$$

It is easy to write the approximate solution as follows if we return back to the original variables:

$$x = \Delta - C_1 e^{-\beta_{eff} t} \left| \sin\left(t - \frac{2\Delta}{\pi C_1 \beta_{eff}} e^{\beta_{eff} t} + C_2 \right) \right| \tag{3.20}$$

The constants C_1 and C_2 are determined through initial conditions.

3.1.3 Comparison between Analytic and Numeric Predictions for the Oscillator Limited from One Side

A comparison between the approximate solution and numeric simulation results for $\Delta = -0.2,\ \beta = 0,\ R = 0.9$ is shown in Fig. 3.4. There is no visible difference between two predictions at least until the oscillations amplitude becomes small. The reason is that the averaging procedure used in this paragraph is valid until the amplitude is much larger than Δ. Otherwise the terms with Δ cannot be considered as small and the system (3.9) or (3.10) is not in the standard form any more.

If the amplitude becomes the same magnitude order as Δ, two totally different effects can be expected depending on the sign of Δ. If Δ is positive, there is no contact between the mass and the limiter in the equilibrium point. That's why their collisions cannot take place as soon as the amplitude becomes smaller than Δ. Then the system's behavior is no different to the linear damped oscillations (or not damped if no damping is assumed in the model, i.e. $\beta = 0$).

The situation is totally different if Δ is negative. Then the process of repeated attenuated collisions starts, which damps the oscillations in a finite time through an infinite number of collisions.

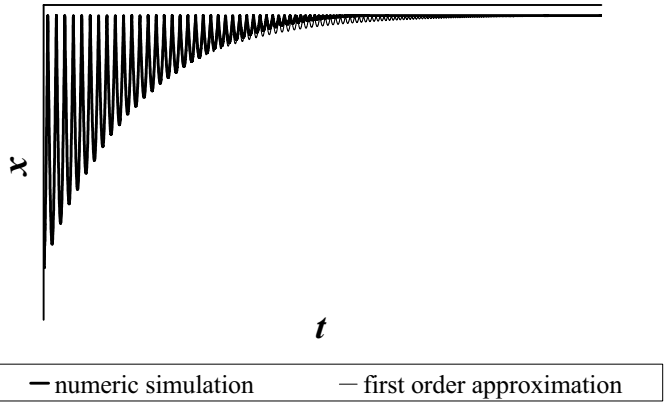

Fig. 3.4. Comparison between the approximate prediction and numeric solution

It is not the objective of this paragraph to discuss the problem of repeated attenuated collisions. It is only sensible to refer here to the book [77], which is devoted to this special problem.

3.1.4 Unfolding Transformation and Averaging for the Free Mass in a Clearance

The second example for an unfolding transformation can be obtained if we analyze a free mass in a clearance (see Fig. 3.5).

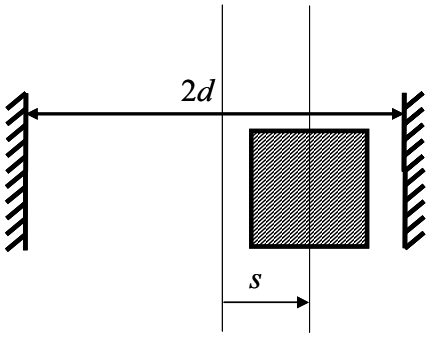

Fig. 3.5. A mass in a clearance

This system can be analyzed under the same assumption the energy dissipation during collisions is small, i.e. the velocity restitution coefficient is close to one. Consider $R=1$ as the unperturbed system. The motion of the mass in the clearance does not differ from a ray between two mirrors with ideal reflection (if we replace time through a space coordinate). In other words it is a straight line, which slope is determined through the initial velocity, folded into the clearance. The basic idea for the further analysis is to unfold this motion back to the straight line.

In order to demonstrate how this idea works, let us consider this system with slightly inelastic collisions. Its motion is governed by equations (3.21). The folded straight line can be easily described by a standardized trigonometric based function (3.22).

$$\begin{aligned} &\ddot{s}=0 \quad \text{if} \quad |s|<d \\ &s_{+}=s_{-}; \quad \dot{s}_{+}=-R\dot{s}_{-} \quad \text{if} \quad |s|=d \end{aligned} \tag{3.21}$$

$$\Pi(z)=\arcsin(\sin z) \tag{3.22}$$

It is sensible to use this function as the unfolding transformation for the problem (3.21):

$$s = \frac{2d}{\pi}\Pi(z) \tag{3.23}$$

The equations of motion in the unfolded variables describe a polygon with small angles between the straight line segments (see Fig. 3.6).

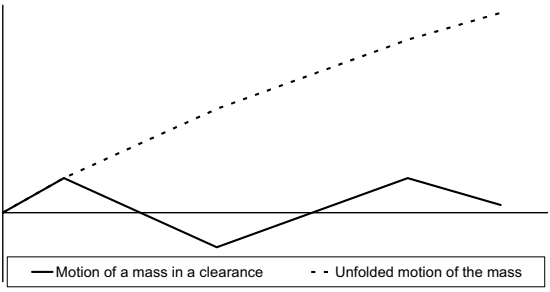

Fig. 3.6. A mass in a clearance: the folded and unfolded motion

Applying transformation (3.23) to (3.21) one obtains the equations governing the unfolded variables.

$$\ddot{z} = 0, \quad \text{if} \quad z \neq \frac{\pi}{2} + \pi n; \qquad \dot{z}_+ = R\dot{z}_-, \quad \text{if} \quad z = \frac{\pi}{2} + \pi n \tag{3.24}$$

These equations can be transformed to the already known form (3.11) if we consider the velocity as the new unknown function and the coordinate z as the new independent variable.

$$\dot{z} = v \tag{3.25}$$

$$\frac{dv}{dz} = 0, \quad z \neq \frac{\pi}{2} + \pi n; \quad v_+ - v_- = -(1-R)v_-, \quad z = \frac{\pi}{2} + \pi n \tag{3.26}$$

The corresponding averaged equation contains only one term describing the velocity reduction due to collisions. It can be solved quite easily, which allows one not only to describe the system in the phase space, but referring back to relationship (3.25), to obtain the general solution in the time domain.

$$\begin{cases} \dfrac{dv}{dz} = -\dfrac{1-R}{\pi}v \\ v = \dfrac{dz}{dt} \end{cases} \Rightarrow \begin{cases} z = \dfrac{\pi}{1-R}\ln(C_1 t + C_2) \\ v = \dfrac{\pi}{1-R}\dfrac{C_1}{C_1 t + C_2} \end{cases} \tag{3.27}$$

Folding this motion back according to the transformation (3.23) the solution in the original variables can be obtained.

$$s = \frac{2d}{\pi}\Pi\left(\frac{\pi}{1-R}\ln\left(C_1 t + C_2\right)\right) \tag{3.28}$$

The free constants C_1 and C_2 are as usual determined through initial conditions. Figure 3.7 displays a comparison between approximate prediction (3.28) and numerical simulation results.

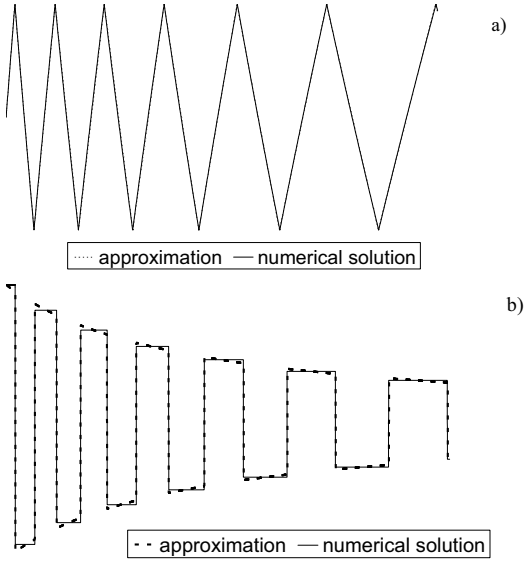

Fig. 3.7. Comparison between asymptotic and numeric solutions; a) coordinates; b) velocities

No difference is visible between two predictions at the coordinate level. In contrast to this the difference on the velocity level is quite clear. This difference shows the main idea for the averaging of systems with slightly inelastic collisions. The vertical lines in Fig. 3.7 b) correspond to the collisions (the velocity changes its sign and its magnitude decreases slightly according to the restitution coefficient). The horizontal lines in the numeric solution occur during the time intervals of the free flight (in these time intervals the velocity does not change). In the approximate solution the situation is totally different. There is no change in the velocity magnitude after each collision. It is the result of our transformation (3.23) based on the ideal reflection. This velocity change is distributed over the whole time interval between collisions and the averaged energy loss is correct. That's why the velocity between the collisions is not constant in the approximate solution.

Summarizing this paragraph one can say that the described ideas help to split analysis into two steps. An unfolding transformation allows eliminating the main part of the discontinuity. It becomes especially effective if the unperturbed un-

folded motion is close to something simple like linear oscillator in case of a mass and a spring near a limit or like constant free motion in case of a mass in a clearance. The second step is averaging. It distributes the small remaining discontinuities over the time interval between collisions. The result is a totally smooth system described by ordinary differential equations without discontinuities at all. In the next paragraphs the effectiveness of this approach is illustrated for the analysis of the classical "mass-on-moving-belt".

3.2. The "Mass-on-Moving-Belt" Limited at One Side: First Order Approximation

The classical mass on moving belt with negative friction gradient is considered in this section. General properties of this system including both pure slip and stick-slip oscillations were discussed in Chapter 2. The effect of almost elastic limit is going to be discussed now. It is easier to start with a one side limit. The system we are going to analyze is shown in Fig. 3.8.

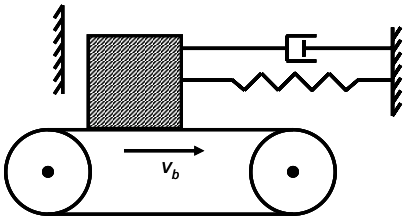

Fig. 3.8. A mass on a belt limited at one side.

It consists of a mass connected to a linear damped spring and placed on a belt moving at constant speed v_b. At one side the motion of the mass is restricted by a limiting element. (The motion is limited from the left side and not from the right side like in the previous section in order to demonstrate the appropriate transformation for this case.) The whole analysis is restricted to pure slip oscillations. The objective is to investigate the influence of the limit as isolated as possible from other effects.

The system is interesting from the technical point of view, because it is the standard and the simplest model for self excited oscillations. In many technical applications the oscillations are nothing dangerous. They become annoying if there is some noise connected with them. There are many different mechanisms by which a self excited mass can emit noise. But periodic collisions with a limiter are one of the strongest possible noise excitation mechanisms. As an example for technical applications hydraulic valves can be mentioned here.

The equation of motion in undimensioned form can be written as follows.

$$\begin{aligned} &s'' + s = -\varepsilon h\left(s'\right) \text{ if } s > \Delta \\ &s_+ = s_-; \quad s'_+ = -Rs'_- \text{ if } s = \Delta \end{aligned} \tag{3.29}$$

The function on the right hand side of (3.29) contains both terms describing the linear spring damping and the nonlinear dry friction with negative friction gradient for small relative velocities.

$$\begin{aligned} &\varepsilon h\left(s'\right) = h_1 s' + h_2 s'^2 + h_3 s'^3 \\ &h_1 = 2\beta - k_1 + 3k_3 v_b{}^2; \; h_2 = -3k_3 v_b; \; h_3 = k_3 \\ &\mu\left(v_r\right) = \mu_s \operatorname{sgn}\left(v_r\right) - k_1 v_r + k_3 v_r{}^3 \end{aligned} \tag{3.30}$$

Parameters h_1, h_2 and h_3 are taken to be small. It is expressed in (3.29) through the formal small parameter ε. We suppose the collisions to be almost elastic, which means the restitution coefficient is close to one. We suppose also the limiting wall is placed close to the position of the mass attached to the unstrained spring.

$$1 - R = O\left(\varepsilon\right); \; \Delta = O\left(\varepsilon\right) \tag{3.31}$$

System (3.29) differs from the considered elementary example (3.4) through friction forces supplying energy to the oscillating mass between the collisions.

The first step is to apply the unfolding transformation expressing the condition, mass cannot penetrate through the limit:

$$s = \Delta + \left|z\right| \tag{3.32}$$

For the unfolded variable the following equations can be easily obtained:

$$\begin{aligned} &z'' + z = -\Delta \operatorname{sgn} z - h_1 z' - h_2 z'^2 \operatorname{sgn} z - h_3 z'^3, \quad \text{if} \quad z \neq 0 \\ &z'_+ - z'_- = -\left(1 - R\right) z'_-, \quad \text{if} \quad z = 0 \end{aligned} \tag{3.33}$$

The second step is to introduce amplitude and phase:

$$z = A \sin\varphi; \quad z' = A \cos\varphi \tag{3.34}$$

Equations (3.35) govern these new variables. The system can be transformed to the form (3.11) if we consider the phase φ as the new independent variable. Restricting analysis to the first order approximation, i.e. neglecting all the terms of higher magnitude orders equation (3.36) can be obtained.

This equation can be easily averaged. The result is an equation containing exactly the same terms, which are already known for the system without limits.

$$
\begin{aligned}
A' &= -\left(\Delta + h_2 A^2 \cos^2\varphi\right)\cos\varphi \operatorname{sgn}\sin\varphi \\
&\quad -\left(h_1 + h_3 A^2 \cos^2\varphi\right) A\cos^2\varphi, \quad \varphi \neq n\pi \\
A_+ - A_- &= -\left(1-R\right)A_-, \quad \varphi = n\pi \\
\varphi' &= 1 + \frac{\Delta + h_2 A^2\cos^2\varphi}{A}\left|\sin\varphi\right| + \left(h_1 + h_3 A^2\cos^2\varphi\right)\cos\varphi\sin\varphi
\end{aligned}
\tag{3.35}
$$

$$
\begin{aligned}
\frac{dA}{d\varphi} &= -\left(\Delta + h_2 A^2\cos^2\varphi\right)\cos\varphi\operatorname{sgn}\sin\varphi + \\
&\quad h_1 A\cos^2\varphi + h_3 A^3\cos^4\varphi, \quad \varphi \neq n\pi \\
A_+ - A_- &= -\left(1-R\right)A_-, \quad \varphi = n\pi
\end{aligned}
\tag{3.36}
$$

The only difference is the effective damping coefficient transformed similarly to (3.18). The corresponding discussion can be found in Chapter 2. The main results of the first order approximation are summarized below.

$$
A_1' = -\frac{1}{2}h_{1eff}A_1 - \frac{3}{8}h_3 A_1^3;\; h_{1eff} = h_1 + \frac{2}{\pi}\left(1-R\right) \tag{3.37}
$$

There are two equilibrium points in (3.37). The first one corresponds to the trivial solution $A_1 = 0$, i.e. to the static equilibrium $z(t) = 0$. The second one is the nontrivial solution:

$$
A_1^* = \sqrt{-\frac{4h_{1eff}}{3h_3}} \tag{3.38}
$$

It corresponds to periodic solutions $z = A_1\sin\left(t+\theta\right)$ with an arbitrary phase θ (the considered system is autonomous). The trivial solution becomes unstable as soon as the expression under the square root in (3.38) becomes positive. It means, for the belt velocities lower than the limit (3.39) the equilibrium point becomes unstable and a stable limit cycle arises, the amplitude of which is given through (3.38).

$$
v_b < v_m\sqrt{1 - \frac{4\beta_{eff}v_m}{3\left(\mu_s - \mu_m\right)}} = v_{b1} \tag{3.39}
$$

This expression for A_1 assumes pure slip, so the increase in amplitude for decreasing v_b will cease when the mass starts sticking to the belt, i.e. when $\max(s') = v_b$. With $\max(s') = A_1$ it is found that sticking first occurs when $A_1 = v_b$. Inserting this into (3.38) and solving for v_b we find that stick occurs first when $v_b < v_{b0}$ where

$$v_{b0} = \sqrt{\tfrac{4}{5}} v_{b1} \tag{3.40}$$

An interesting and unexpected result here is the fact, that the stationary amplitude of the oscillator limited at one side does not depend on the parameter Δ at least according to the first order approximation. The influence of Δ on the oscillation frequency can be easily found if we average the equation for φ in (3.35). The result is

$$\varphi_1' = \omega_1 = 1 + \frac{2}{\pi A_1}\left(\Delta + \frac{1}{3} h_2 A_1^{\,2}\right) \tag{3.41}$$

It means that the oscillation frequency depends linearly on the distance between the equilibrium point and the limiter. It is necessary to notice at this point that this is the frequency for the unfolded system. The main frequency for the original variables is certainly twice as much as ω_1. It is also important to notice, that the frequency depends on Δ in the small terms only, so one could expect a similar dependency for the amplitude. In order to investigate this question the second order approximation to (3.36) is necessary.

3.3. Second Order Approximation in Systems with Almost Elastic Collisions

3.3.1 General Mathematical Approach

The second order approximation for a system in form (3.11) can be easily obtained if we apply the general idea of almost identical transformations (*cf.* paragraph 1.2.2 for details). Consider a system where the second order terms are taken into account explicitly:

$$\begin{aligned} x' &= \varepsilon X_1(x,t) + \varepsilon^2 X_2(x,t), \quad t \neq n\pi \\ x_+ - x_- &= -\varepsilon f_1(x_-) - \varepsilon^2 f_2(x_-), \quad t = n\pi \end{aligned} \tag{3.42}$$

The basic idea is to perform an almost identical variable transformation as follows:

$$x = \xi + \varepsilon u_1(\xi,t) + \varepsilon^2 u_2(\xi,t) + \ldots \tag{3.43}$$

The aim is to obtain an autonomous equation for the new variable ξ, which does not contain time:

$$\xi' = \varepsilon \Xi_1(\xi) + \varepsilon^2 \Xi_2(\xi) + \ldots \tag{3.44}$$

We apply (3.43) to (3.42), take (3.44) into account and balance the terms with the same powers of the small parameter:

$$\begin{aligned} &\varepsilon \Xi_1 + \varepsilon \frac{\partial u_1}{\partial t} + \varepsilon^2 \Xi_2 + \varepsilon^2 \frac{\partial u_1}{\partial \xi} \Xi_1 + \varepsilon^2 \frac{\partial u_2}{\partial t} + \ldots = \\ &\qquad \varepsilon X_1(\xi,t) + \varepsilon^2 X_2(\xi,t) + \varepsilon^2 \frac{\partial X_1}{\partial \xi} u_1 + \ldots, t \neq n\pi \\ &\varepsilon u_{1+} + \varepsilon^2 u_{2+} + \ldots - \varepsilon u_{1-} - \varepsilon^2 u_{2-} - \ldots = \\ &\qquad -\varepsilon f_1(\xi) - \varepsilon^2 \frac{\partial f_1}{\partial \xi} u_{1-} - \varepsilon^2 f_2(\xi) \ldots, \quad t = n\pi \end{aligned} \tag{3.45}$$

For the unknown functions u_1 and u_2 we obtain the following equations:

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= X_1(\xi,t) - \Xi_1, \quad t \neq n\pi \\ u_{1+} - u_{1-} &= -f_1(\xi), \quad t = n\pi \end{aligned} \tag{3.46}$$

$$\begin{aligned} \frac{\partial u_2}{\partial t} &= X_2(\xi,t) + \frac{\partial X_1}{\partial \xi} u_1 - \Xi_2 - \frac{\partial u_1}{\partial \xi} \Xi_1 \\ u_{2+} - u_{2-} &= -\frac{\partial f_1}{\partial \xi} u_{1-}, \quad t = n\pi \end{aligned} \tag{3.47}$$

Functions u_1 and u_2 have to remain limited with respect to time otherwise we could not balance the terms with different powers of the small parameter. In order

to fulfill this condition we must assure that the average change of the right hand sides of our systems vanish. This determines the functions Ξ_1 and Ξ_2 :

$$\Xi_1 = \frac{1}{2\pi}\int_0^{2\pi} X_1(\xi,t)\,dt - \frac{1}{\pi} f_1(\xi) \tag{3.48}$$

$$\Xi_2 = \frac{1}{2\pi}\int_0^{2\pi}\left(X_2(\xi,t) + \frac{\partial X_1}{\partial \xi} u_1 - \frac{\partial u_1}{\partial \xi}\Xi_1 \right) dt - \frac{1}{2\pi}\frac{\partial f_1}{\partial \xi}\left(u_{1-}(0) + u_{1-}(\pi)\right) - \frac{1}{\pi} f_2(\xi) \tag{3.49}$$

In order to calculate Ξ_2 we have to determine the function u_1. It can be done in different ways, because according to (3.46) this function depends on an arbitrary slow function $C(\xi)$.

$$\begin{aligned} u_1 &= \int_0^t \left(X_1(\xi,t) - \Xi_1(\xi) \right) dt + C(\xi), \quad t \neq n\pi \\ u_{1+} &= u_{1-} - \varepsilon f_1(\xi), \quad t = n\pi \end{aligned} \tag{3.50}$$

The most useful way is to require that the average of the function u_1 between the discontinuities must be equal to zero.

$$C = -\int_0^{2\pi}\left(\int_0^t \left(X_1(\xi,\tau) - \Xi_1 \right) d\tau \right) dt \tag{3.51}$$

If C is chosen according to this relationship, then the last term under the integral (3.49) vanishes and we obtain finally:

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial u_1}{\partial \xi}\Xi_1 dt = \frac{1}{2\pi}\Xi_1\int_0^{2\pi}\frac{\partial u_1}{\partial \xi} dt = \frac{1}{2\pi}\Xi_1\frac{\partial}{\partial \xi}\left(\int_0^{2\pi} u_1 dt\right) = 0 \\ &\Xi_2 = \frac{1}{2\pi}\int_0^{2\pi}\left(X_2 + \frac{\partial X_1}{\partial \xi} u_1 \right) dt - \frac{1}{2\pi}\frac{\partial f_1}{\partial \xi}\left(u_{1-}(0) + u_{1-}(\pi)\right) - \frac{1}{\pi} f_2 \end{aligned} \tag{3.52}$$

The accuracy of the system (3.44) can be proved as usual (*cf.* Appendix V). Taking further terms into account (which may be quite elaborated) one can get an approximation of any required order.

3.3.2 The Second Order Approximation for the Amplitude of the Mass on Moving Belt Limited from One Side

The described approach can be used to obtain the second approximation for the "mass on moving belt" and investigate how the oscillations amplitude depends on Δ. The first step is to rewrite (3.36) holding not only the first order but also the second order terms:

$$\begin{aligned} A' &= \varepsilon X_1(A,\varphi) + \varepsilon^2 X_2(A,\varphi), \quad \varphi \neq n\pi \\ A_+ - A_- &= -\varepsilon f(A_-), \quad \varphi = n\pi \end{aligned} \tag{3.53}$$

The following symbols are used here:

$$\begin{aligned} \varepsilon X_1(A,\varphi) = -\left(\Delta + h_2 A^2 \cos^2\varphi\right)\cos\varphi \operatorname{sgn}\sin\varphi - \\ h_1 A \cos^2\varphi - h_3 A^3 \cos^4\varphi \end{aligned} \tag{3.54}$$

$$\begin{aligned} \varepsilon^2 X_2(A,\varphi) = -\frac{\varepsilon X_1}{A}\left(\Delta + h_2 A^2 \cos^2\varphi\right)\left|\sin\varphi\right| - \\ \frac{\varepsilon X_1}{A}\left(h_1 A\cos\varphi + h_3 A^3 \cos^3\varphi\right)\sin\varphi \\ \varepsilon f(A) = (1-R)A \end{aligned} \tag{3.55}$$

The average of (3.54) is already known (*cf.* (3.37)). IN order to obtain the second order approximation, the function Ξ_2 must be calculated according to (3.52):

Function X_2 can be averaged quite easily:

$$\left\langle \varepsilon^2 X_2(A,\varphi)\right\rangle = \frac{4}{3\pi}\Delta h_1 + \frac{4}{5\pi}\left(\Delta h_3 + h_1 h_2\right)A^2 + \frac{4}{7\pi}h_2 h_3 A^4 \tag{3.56}$$

However it is not sufficient in order to obtain the second order approximation. In addition to (3.56) it is necessary to take the small oscillating terms into account, which describe the difference between the solutions of the original equation and the first order approximation up to the magnitude order $O(\varepsilon)$. For that purpose it is necessary to calculate the correcting function u:

$$\varepsilon\frac{\partial u_1}{\partial\varphi}=\left(X_1(A,\varphi)-\langle X_1(A,\varphi)\rangle_\varphi-\frac{1}{\pi}f(A)\right)$$
$$=-\Delta\cos\varphi\,\mathrm{sgn}\sin\varphi-\frac{1}{4}h_2A^2\left(\cos3\varphi+3\cos\varphi\right)\mathrm{sgn}\sin\varphi$$
$$-\frac{1}{4}h_3A^3\left(2\cos2\varphi+\frac{1}{2}\cos4\varphi\right)-\frac{1}{2}h_1A\cos2\varphi+\frac{1-R}{\pi}A \qquad (3.57)$$
$$\varepsilon u_1=C-\Delta\sin\varphi-\frac{1}{4}h_1A\sin2\varphi-\frac{1}{4}h_2A^2\left(\frac{1}{3}\sin3\varphi+3\sin\varphi\right)$$
$$-\frac{1}{4}h_3A^3\left(\sin2\varphi+\frac{1}{8}\sin4\varphi\right)+\frac{1-R}{\pi}A\varphi\ \text{ for } 0<\varphi<\pi$$

The free constant C is determined through the requirement that the average of the correcting function must be equal to zero:

$$C=-\int_0^{2\pi}\left(\int_0^t\left(X_1(\xi,\tau)-\Xi_1\right)d\tau\right)dt=\frac{2}{\pi}\Delta+\frac{14}{9\pi}h_2A^2-\frac{1-R}{2}A \qquad (3.58)$$

Now the second term in (3.52) can be calculated:

$$\varepsilon\frac{\partial X_1}{\partial A}=-h_1\cos^2\varphi-2h_2A\cos^3\varphi\,\mathrm{sgn}\sin\varphi-3h_3A^2\cos^4\varphi$$
$$\left\langle\varepsilon^2\frac{\partial X_1}{\partial A}u_1\right\rangle=\frac{1-R}{\pi^2}\frac{28}{9}h_2A^2-\frac{\Delta}{\pi}\left(\frac{h_1}{3}+\frac{21}{20}h_3A^2\right)+\frac{1}{5\pi}h_1h_2A^2 \qquad (3.59)$$
$$-\frac{31}{140\pi}h_2h_3A^4$$

Finally, all the terms together give us the equation of the second approximation:

$$A_2'=-\frac{h_1A_2}{2}-\frac{3h_3A_2^{\,3}}{8}-\frac{(1-R)}{\pi}A_2+\frac{\Delta}{\pi}\left(h_1-\frac{h_3A_2^{\,2}}{4}-2A_2\frac{1-R}{\pi}\right)$$
$$+\frac{h_1h_2A_2^{\,2}}{\pi}+\frac{7h_2h_3A_2^{\,4}}{20\pi}-\frac{1-R}{\pi}A_2^{\,2}\left(\frac{1-R}{2}+\frac{14}{9\pi}h_2\left(A_2-2\right)\right) \qquad (3.60)$$

The stationary solution can be obtained asymptotically if we set the right hand side of this equation to zero. (A_2^* is the stationary amplitude according to the second approximation.)

$$A_2^* = A_1^* + \varepsilon \tilde{A}_2; \quad \tilde{A}_2 = -\frac{\Xi_2\left(A_1^*\right)}{\left.\frac{d\Xi_1}{dA}\right|_{A_1^*}}; \tag{3.61}$$

Here according to (3.60) the following symbols are used:

$$\left.\frac{d\Xi_1}{dA}\right|_{A_1^*} = -\frac{1}{2}h_1 - \frac{3}{4}h_3\left(A_1^*\right)^2 - \frac{(1-R)}{\pi}$$

$$\Xi_2 = \frac{\Delta}{\pi}\left(h_1 - \frac{h_3 A^2}{4} - 2A\frac{1-R}{\pi}\right) + \frac{h_1 h_2 A^2}{\pi} + \frac{7h_2 h_3 A^4}{20\pi} \tag{3.62}$$

$$-\frac{1-R}{\pi}A^2\left(\frac{1-R}{2} + \frac{14}{9\pi}h_2(A-2)\right)$$

3.3.3 Discussion of the Results and Comparisons with Numeric Experiments

It is obvious that the stationary amplitude depends linearly from the distance between the (unstable) equilibrium and the limiter. This conclusion can be easily compared with numeric simulation results. Fig. 3.9 shows both predictions for the following parameter values: $h_1 = -0.05; h_2 = -0.03; h_3 = 0.01; R = 0.95$.

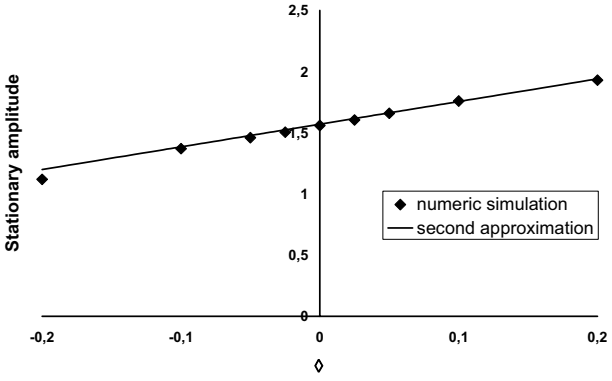

Fig. 3.9. Stationary amplitude as a function of the distance from the (unstable) equilibrium of the mass on moving belt to the limit: A comparison between numeric simulation and analytic prediction

The prediction gives quite accurate results for $-0.2 < \Delta < 0.2$, which is a rather wide range for a small parameter.

Summarizing the results concerning the "mass on moving belt" limited at one side one could give the following qualitative explanation for the obtained results. The system's motion in the considered case can be unfolded to an almost harmonic oscillation (sinus-function). The effect of collisions, for the first order accuracy, can be reduced to an additional effective damping. The energy supplied to the mass from the belt due to the negative friction gradient results in the increasing oscillation amplitude until the nonlinear terms in the increasing friction at high velocities compensate this effect. The system's behaviour is similar to an oscillator without limiter. The nonlinear effect of collisions becomes visible first in the second order approximation. It means the collisions alone are not able to limit the oscillations amplitude. They can only increase the minimal negative friction gradient necessary for the self-excitation. A system with much stronger nonlinear effect of the collisions is analyzed in the next paragraph.

3.4. The "Mass on Moving Belt" in a clearance

The same classical mass on the moving belt with negative friction gradient as in section 3.2 is considered in this paragraph. It is supposed that the mass is limited from both sides. We are going to show that such a constraint changes all the dynamical properties of the system totally. The system we are going to analyze is shown in Fig. 3.10.

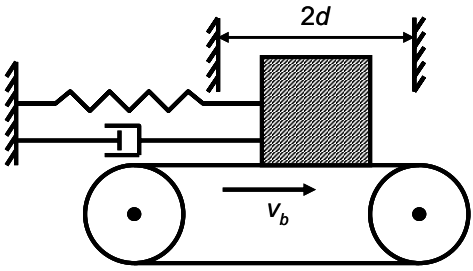

Fig. 3.10. A mass on a belt in a clearance

It consists of a mass connected to a linear damped spring and placed on a belt moving at a constant velocity v_b. The motion of the mass is restricted from both sides by limiting elements. The whole analysis is restricted to pure slip oscillations. The objective is to investigate the influence of the clearance as isolated as possible from other effects.

The system is interesting from the technical point of view, because clearances are very usual in applications. Generally small clearances are necessary for assembling, otherwise it would be impossible to put parts together and assure that

they can move relative to each other while operating. Typical examples of such systems are: clearances between gears; motions, perpendicular to the flow, of hydraulic valves in pipes. In both cases self excitation mechanisms can occur. In gears the self excitation comes usually from the non-conservative friction forces (not necessarily due to negative friction gradient). In hydraulic valves the excitation mechanism is a sort of flutter. In any case Fig. 3.10 shows the simplest model containing both a self excitation mechanism and a clearance. Another interesting application area is dynamic clearance for example in systems limited by preloaded springs with dry friction. These technical applications should be discussed separately. The analysis in this section is restricted to the simplest system containing both a self excitation mechanism and a clearance.

3.4.1 The Governing Equations and the Unfolding Transformation

The equation of motion for our system in undimensioned form can be written as follows:

$$\begin{aligned} &s'' + s = -\varepsilon h(s'), \quad |s| < \Delta \\ &s_+ = s_-; \quad s'_+ = -Rs'_-, \quad |s| = \Delta \end{aligned} \tag{3.63}$$

Function $h(s')$ here contains the terms, which describe the nonlinear friction characteristic with negative slope:

$$\begin{aligned} \varepsilon h(s') &= \left(2\beta - k_1 + 3k_3 v_b^{\,2}\right)s' - 3k_3 v_b s'^2 + k_3 s'^3 \\ &= h_1 s' + h_2 s'^2 + h_3 s'^3 \\ h_1 &< 0 \end{aligned} \tag{3.64}$$

We assume here that the equilibrium point of the mass on the moving belt is in the middle of the clearance, but we do not assume that the clearance is small. System (3.63) can be regularized by means of the unfolding transformation (3.23).

$$\begin{aligned} &s = \frac{2\Delta}{\pi}\Pi(z) \Rightarrow \\ &s' = \frac{2\Delta}{\pi}M(z)z'; s'' = \frac{2\Delta}{\pi}M(z)z'' \text{ for } z \neq \frac{\pi}{2} + \pi n \\ &M(z) = \operatorname{sgn}\cos z \end{aligned} \tag{3.65}$$

Applying this transformation to (3.63) one can easily obtain equations for the unfolded variable z :

$$z'' + M(z)\Pi(z) = -\bar{h}_1 z' - \bar{h}_2 z'^2 M(z) - \bar{h}_3 z'^3, \quad z \neq \frac{\pi}{2} + \pi n$$

$$z'_+ - z'_- = -(1-R)z'_-, \quad z = \frac{\pi}{2} + \pi n \tag{3.66}$$

$$\bar{h}_1 = h_1; \ \bar{h}_2 = \frac{2\Delta}{\pi} h_2; \ \bar{h}_3 = \frac{4\Delta^2}{\pi^2} h_3$$

We assume that the slope of the friction characteristics (determining energy input) and energy dissipation (both due to friction and collisions) are small values of the same magnitude order, i.e.

$$\bar{h}_1 = O(\varepsilon); \bar{h}_2 = O(\varepsilon); \bar{h}_3 = O(\varepsilon); (1-R) = O(\varepsilon); \Delta = O(1) \tag{3.67}$$

3.4.2 Analyzing the Unperturbed System and Introducing Energy as the Slow Variable

The unperturbed system related to (3.66) must be investigated in order to find a transformation converting (3.66) to the "standard form" for averaging in systems with slightly inelastic collisions (3.11). Our objective is (according to standard procedure) to find its first integrals and to take them as new unknown functions which will be then automatically slow. In this case, the unperturbed system is a conservative one (we have supposed both energy input and dissipation are small):

$$z'' + M(z)\Pi(z) = 0 \tag{3.68}$$

It has an integral corresponding to the energy conservation law:

$$E = \frac{1}{2} z'^2 + Q(z); \quad Q(z) = \int_0^z M(\varsigma)\Pi(\varsigma) d\varsigma \tag{3.69}$$

Here E is the integration constant which is equal to the full energy of the unperturbed system, $Q(z)$ is the corresponding potential energy. The introduced functions are shown in Fig. 3.11.

The sense of the transformation (3.23) was to unfold the motion of the mass to a "straight line". It means the variable z have to increase monotonously. As a consequence we can limit the following analysis to the rotation of the equivalent pendulum omitting the oscillation case completely. It means the full energy must be larger than the maximum of the potential energy:

$$E > \frac{\pi^2}{8} \tag{3.70}$$

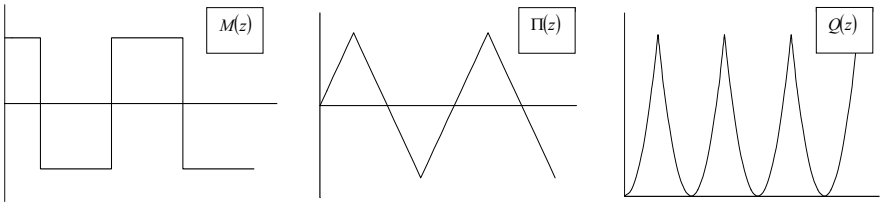

Fig. 3.11. Functions $M(z)$, $\Pi(z)$, $Q(z)$.

This assumption means from the physical point of view that the mass does not change the direction of motion between collisions. Another limit is given by the condition, that we consider pure slip motions. This kind of motion is possible only for

$$s' < v_b \Leftrightarrow E < \frac{\pi^2 {v_b}^2}{8\Delta^2} \tag{3.71}$$

Qualifying our analysis in this way we can solve (3.69) with respect to the pendulum velocity:

$$z' = \sqrt{2\left(E - Q(z)\right)} \tag{3.72}$$

The next step is to apply transformation (3.69) to the full perturbed system (3.66) considering the full energy as the new unknown function:

$$\begin{aligned} E' &= z'z'' + M(z)\Pi(z)z' = -\bar{h}_1 z'^2 - \bar{h}_2 z'^3 - \bar{h}_3 z'^4, z \neq \frac{\pi}{2} + \pi n \\ E_+ - E_- &= -\left(1 - R^2\right)\left(E_- - \frac{\pi^2}{8}\right), \quad z = \frac{\pi}{2} + \pi n \end{aligned} \tag{3.73}$$

Considering z according to (3.72) as the new independent variable the following equation in the "standardized form" can be obtained:

$$\frac{dE}{dz} = -\bar{h}_1\sqrt{2\left(E-Q(z)\right)} - 2\bar{h}_2\left(E-Q(z)\right)M(z) - \bar{h}_3\left(2\left(E-Q(z)\right)\right)^{\frac{3}{2}}, \quad z \neq \frac{\pi}{2}+\pi n \tag{3.74}$$

$$E_+ - E_- = -\left(1-R^2\right)\left(E_- - \frac{\pi^2}{8}\right), \quad z = \frac{\pi}{2}+\pi n$$

The following integrals must be calculated in order to average system (3.74):

$$J_1 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\sqrt{2E-z^2}\,dz = \frac{1}{2}\sqrt{2E-\frac{\pi^2}{4}} + \frac{2}{\pi}E\arcsin\left(\frac{\pi}{2\sqrt{2E}}\right)$$

$$J_2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sqrt{E-Q(z)}\,\operatorname{sgn}\cos z\,dz = 0$$

$$J_3 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left(2E-z^2\right)^{3/2}dz = \left(2E-\frac{\pi^2}{4}\right)^{3/2} + \frac{3E}{4}\sqrt{2E-\frac{\pi^2}{4}} + \frac{3}{\pi}E^2\arcsin\left(\frac{\pi}{2\sqrt{2E}}\right) \tag{3.75}$$

$$J_4 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{dz}{\sqrt{2E-z^2}} = 2\arcsin\left(\frac{\pi}{2\sqrt{2E}}\right)$$

The first order approximation to (3.74) is:

$$\frac{dE}{dz} = -\bar{h}_1\left\{\frac{1}{2}\sqrt{2E-\frac{\pi^2}{4}} + \frac{2}{\pi}E\arcsin\left(\frac{\pi}{2\sqrt{2E}}\right)\right\} - \frac{1-R^2}{\pi}\left(E-\frac{\pi^2}{8}\right) - \bar{h}_3\left\{\left(2E-\frac{\pi^2}{4}\right)^{\frac{3}{2}} + \frac{3}{4}E\sqrt{2E-\frac{\pi^2}{4}} + \frac{3}{\pi}E^2\arcsin\left(\frac{\pi}{2\sqrt{2E}}\right)\right\} \tag{3.76}$$

3.4.3 Discussion of the Results

Stationary solution of this equation determines periodic oscillations of the mass on moving belt with two collisions per oscillation period. The physical meaning of this solution becomes clear if we consider the particular case $h_3 = 0$, i.e. friction always decreases with the velocity. Then the mass gets additional energy during slipping and dissipates while colliding. If the energy obtained during slipping is larger than the energy lost by impact, the total energy will increase. But the oscillation amplitude is restricted by the fixed clearance length. So the only way to increase the energy is to increase the velocity and the corresponding oscillation frequency. In other words not the amplitude but the frequency characterizes the stationary regime sensibly. The corresponding transient simulation results are shown in Fig. 3.12.

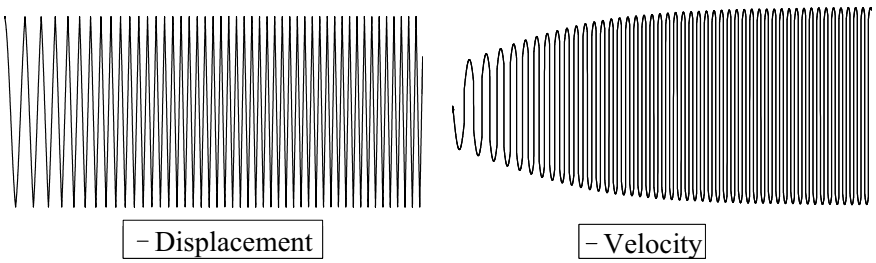

Fig. 3.12. Transient motions of the mass on the belt in a clearance

Is the stationary energy level E_* found from equation (3.76), the corresponding stationary frequency can be found from (3.72). The energy deviation from its stationary level has the magnitude order of the small parameter.

$$\frac{dz}{dt} = \sqrt{2\left(E_* - Q(z)\right)} \quad \Rightarrow \quad t = t_0 + \int_0^z \frac{d\varsigma}{\sqrt{2\left(E_* - Q(\varsigma)\right)}} \tag{3.77}$$

These relationships are valid only between collisions and allow calculating the oscillation period and the corresponding frequency:

$$T = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\varsigma}{\sqrt{2\left(E_* - Q(\varsigma)\right)}} = 4\arcsin\left(\frac{\pi}{2\sqrt{2E_*}}\right)$$
$$\omega = \frac{\pi}{2\arcsin\left(\frac{\pi}{2\sqrt{2E_*}}\right)} \tag{3.78}$$

$$\bar{h}_1\left\{\frac{1}{2}\sqrt{2E_*-\frac{\pi^2}{4}}+\frac{2}{\pi}E_*\arcsin\left(\frac{\pi}{2\sqrt{2E_*}}\right)\right\}=\frac{1-R^2}{\pi}\left(\frac{\pi^2}{8}-E_*\right)$$
$$-\bar{h}_3\left\{\left(2E_*-\frac{\pi^2}{4}\right)^{\frac{3}{2}}+\frac{3}{4}E_*\sqrt{2E_*-\frac{\pi^2}{4}}+\frac{3}{\pi}E_*{}^2\arcsin\left(\frac{\pi}{2\sqrt{2E_*}}\right)\right\} \quad (3.79)$$

This relationship together with the equation for the stationary energy level (3.79) can be considered as a system of two parametric equations determining how the oscillation frequency depends on any physical parameter, whereas the energy E_* is a formal parameter. Figure 3.13 shows for example how the frequency depends on the parameter h_1 for $h_3 = 0$. It also shows the comparison with the numeric solution, which seems to be quite acceptable in this case.

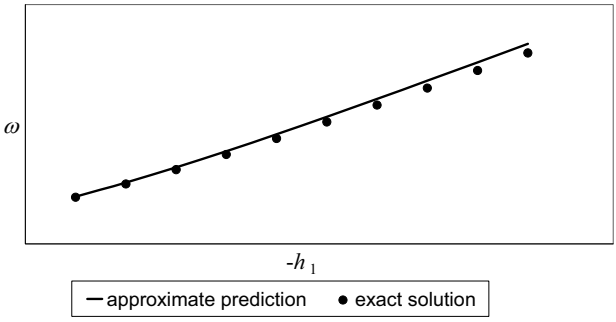

Fig. 3.13. Comparison between numeric solution and analytic approximation for $R = 0.95$

The maximal belt velocity limiting the area of self-excitation and the corresponding frequency are given by (3.80).

$$v_b{}^{\max}=\sqrt{\frac{k_1-2\beta}{3k_3}-\frac{\Delta^2}{4}};\quad \omega\left(v_b{}^{\max}\right)=1 \quad (3.80)$$

The minimal velocity for pure slip oscillations is given by (3.71). The corresponding frequency is

$$\omega\left(v_b{}^{\min}\right)=\frac{\pi}{2\arcsin\left(\dfrac{\Delta}{v_b{}^{\min}}\right)} \quad (3.81)$$

Motions of the considered type are possible if inequalities (3.70) and (3.71) are compatible: $v_b > \Delta$. Figure 3.14 shows how the frequency depends on the belt velocity for the following parameter values:
$2\beta - k_1 = -1;\ k_3 = 0{,}5;\ R = 0{,}95;\ \Delta = 0{,}5$.

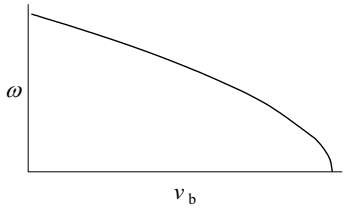

Fig. 3.15. Oscillation frequency as a function of the belt velocity

The fulfilled analysis underlines the qualitative difference between the system limited at one side and the same system in a clearance. The last one is principally nonlinear. A stable limit cycle is possible without any additional assumptions concerning the nonlinearity of friction. Collisions are able to stabilize the limit cycle themselves. The system can be interpreted as a frequency transformer. It is possible to control the oscillation frequency varying the belt velocity and as a result changing the negative slope of the friction characteristics. The same effect can be achieved if it is possible to change the clearance length. These properties of the mass on the belt in a clearance open an opportunity to use it or similar systems as a dynamic damper which frequency can be easily adjusted in a wide range. It can be also used in order to excite vibrations in machines if the oscillation frequency must be changed while operating.

3.5. Resonance of the Impact Oscillator Limited at One Side under External Excitation

Self excited systems with almost elastic collisions were considered in the previous sections. Now we are going to investigate how discontinuous systems behave if they are excited by some external source. We start with the simplest example – a linear oscillator limited at one side.

3.5.1 Equations of Motion and the Unfolding Transformation

Fig. 3.15 shows the system under consideration: a mass described by its coordinate $s(t)$ is attached to a damped linear spring and excited by a harmonic force. The mass can collide with a quasi-elastic stop at a distance Δ from its static

equilibrium point. As usual in this chapter, the analysis is restricted to oscillations with collisions only.

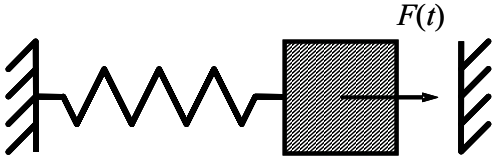

Fig. 3.15. The classical oscillator limited at one side and excited by an external force

The equation of motion and corresponding kinematic conditions describing collisions can be written in an undimensioned form as follows:

$$\begin{aligned} &\ddot{s}+\beta\dot{s}+s=\varepsilon\sin\omega t, \quad \text{if} \quad s<\Delta \\ &s_{+}=s_{-}; \quad \dot{s}_{+}=-R\dot{s}_{-}, \quad \text{if} \quad s=\Delta \end{aligned} \tag{3.82}$$

This system differs from the already considered system (3.4) through the external excitation term $\varepsilon\sin\omega t$. Damping coefficient is supposed to be positive and the motion is limited from the right side. The already known unfolding transformation (3.4) can be applied here too:

$$s=\Delta-|z| \tag{3.83}$$

Applying this transformation to the equations (3.82) one can easily obtain equations governing the unfolded variable:

$$\begin{aligned} &\ddot{z}+\beta\dot{z}+z=\left(\Delta-\varepsilon\sin\omega t\right)\operatorname{sgn} z, \quad z\neq 0 \\ &\dot{z}_{+}-\dot{z}_{-}=-\left(1-R\right)\dot{z}_{-}, \quad z=0 \end{aligned} \tag{3.84}$$

We suppose here, as usual in the resonant problems, that damping, dissipation due to collisions and amplitude of the excitation force are small. In addition we assume the distance between the mass equilibrium point and the limit to be also small (*cf.* (3.8)). The unperturbed system corresponding to (3.84) is a linear conservative oscillator. So the Van-der-Pol transformation can be applied to transform (3.84) to an appropriate form:

$$z=A\sin\varphi; \quad \dot{z}=A\cos\varphi \tag{3.85}$$

It is also useful to introduce a new uniformly rotating phase instead of time

$$\psi=\omega t \tag{3.86}$$

and to convert to an autonomous system:

$$\begin{aligned}
&\dot{A} = -\beta A \cos^2\varphi + \left(\Delta - \varepsilon \sin\psi\right)\cos\varphi \operatorname{sgn}\sin\varphi, \;\; \varphi \neq n\pi \\
&A_+ - A_- = -\left(1-R\right)A_-, \;\; \varphi = n\pi \\
&\dot{\varphi} = 1 + \beta \sin\varphi\cos\varphi - \frac{\Delta - \varepsilon\sin\psi}{A}\left|\sin\varphi\right| \\
&\dot{\psi} = \omega
\end{aligned} \tag{3.87}$$

All the previous problems discussed in this chapter were described by autonomous systems of the second order, i.e. their phase space was 2-dimensional. Now we consider the first problem with 3-dimensional phase space. System (3.87) is still not converted to the "standard" form for averaging (3.11), because it contains two fast rotating phases φ and ψ.

3.5.2 Resonances in the Almost Linear System

Different types of solutions can be investigated in this system. Usually the resonant solutions are of the most interest in applications, because in these cases the oscillation amplitude is large (has the magnitude order of one) even though the excitation force is small. From the physical point of view the small energy input due to harmonic excitation force is balanced in the resonant case by the small energy dissipation due to internal damping and almost elastic collisions. But system (3.84) is nonlinear not only due to collisions but also because the excitation term on the right hand side is multiplied by $\operatorname{sgn} z$. What are the resonances in such a nonlinear system?

The general definition of a resonance in the averaging method is well known (see for example [114, 139]). Especially simple is this definition for almost linear systems like (3.87), where all nonlinearities are concentrated in the small terms.

We will understand a parameter combination for which the time averages of the right hand sides of the full equations become discontinuous as *a resonance surface*.

The resonant solutions are solutions of our system for parameter combinations being in the vicinity of the resonant surface (in sense of the small parameter).

This definition seems to be very unclear for an inexperienced reader. We are going to illustrate it considering system (3.87). The corresponding unperturbed system is quite easy:

$$\dot{A} = 0; \;\; \dot{\varphi} = 1; \;\; \dot{\psi} = \omega \tag{3.88}$$

Its solution is

$$A = A_0; \;\; \varphi = t + \varphi_0; \;\; \psi = \omega t + \psi_0 \tag{3.89}$$

Now we have to average the right hand sides of (3.87) along the solutions (3.89) and find for which values of the parameter ω this average becomes discontinuous. The only terms which can become discontinuous are

$$\begin{aligned}
&\left\langle \sin\psi \cos\varphi \operatorname{sgn}\sin\varphi \right\rangle = \frac{1}{2\pi}\int_0^{2\pi} \sin\left(\omega t+\gamma\right)\cos t \operatorname{sgn}\sin t dt \\
&\left\langle \sin\psi \left|\sin\varphi\right| \right\rangle = \frac{1}{2\pi}\int_0^{2\pi} \sin\left(\omega t+\gamma\right)\left|\sin t\right| dt \\
&\gamma = \psi_0 - \varphi_0
\end{aligned} \tag{3.90}$$

It is easy to notice considering Fig. 3.2 that the function $\left|\sin t\right|$ is π-periodic, in other words its frequency is 2. Even more, if we take a Fourier-expansion for this function it will contain all the components with frequencies $2k$, $k=1,2,3,\ldots$ and only these components. But the terms we are interested in are nothing different but the Fourier-coefficients for this function! In other words the integrals (3.90) are equal to zero for all values of the excitation frequency ω except the countable number of values:

$$\omega_k = 2k;\ \ k=1,2,3,\ldots \tag{3.91}$$

These are our resonant parameter values. Let us investigate equations (3.87) in the small vicinity of each resonance.

3.5.3 Averaging in the Vicinities of the Almost Linear Resonances

We introduce a small frequency delay and a new variable as follows:

$$\delta = \frac{\omega}{2k} - 1;\ \ \theta = \varphi - \frac{\psi}{2k} \tag{3.92}$$

Now our equations can be rewritten in the following form:

$$\begin{aligned}
&\dot{A} = -\beta A\cos^2\varphi + \Delta\cos\varphi\operatorname{sgn}\sin\varphi \\
&\quad -\varepsilon\sin 2k\left(\varphi-\theta\right)\cos\varphi\operatorname{sgn}\sin\varphi,\ \ \varphi\neq n\pi \\
&A_+ - A_- = -\left(1-R\right)A_-,\ \ \varphi = n\pi \\
&\dot{\theta} = -\delta + \beta\sin\varphi\cos\varphi - A^{-1}\left(\Delta-\varepsilon\sin 2k\left(\varphi-\theta\right)\right)\left|\sin\varphi\right| \\
&\dot{\varphi} = 1 + \beta\sin\varphi\cos\varphi - A^{-1}\left(\Delta-\varepsilon\sin 2k\left(\varphi-\theta\right)\right)\left|\sin\varphi\right|
\end{aligned} \tag{3.93}$$

Supposing δ to be small we can consider (3.93) as a system in the required form. It contains two slow variables, one of which is discontinuous, and one fast rotating phase. The discontinuous averaging procedure can be applied to this system. In order to do it the following integrals must be calculated:

$$\begin{aligned} J_1 &= \frac{1}{2\pi}\int_0^{2\pi} \sin 2k\varphi \left|\sin\varphi\right| d\varphi = \frac{1}{\pi}\int_0^{\pi} \sin 2k\varphi \sin\varphi d\varphi = 0 \\ J_2 &= \frac{1}{2\pi}\int_0^{2\pi} \cos 2k\varphi \left|\sin\varphi\right| d\varphi = -\frac{2}{\pi\left(4k^2-1\right)} \\ J_3 &= \frac{1}{2\pi}\int_0^{2\pi} \sin 2k\varphi \cos\varphi \operatorname{sgn}\sin\varphi d\varphi = \frac{4k}{\pi\left(4k^2-1\right)} \\ J_4 &= \frac{1}{2\pi}\int_0^{2\pi} \cos 2k\varphi \cos\varphi \operatorname{sgn}\sin\varphi d\varphi = 0 \end{aligned} \tag{3.94}$$

Averaging (3.93) and taking (3.94) into account the following equations of the first order approximation can be obtained:

$$\begin{aligned} \dot{A} &= -\left(\frac{1}{2}\beta + \frac{1-R}{\pi}\right)A - \frac{4k\varepsilon}{3\pi\left(4k^2-1\right)}\cos 2k\theta \\ \dot{\theta} &= -\delta - \frac{2\Delta}{\pi A} + \frac{2\varepsilon}{3\pi\left(4k^2-1\right)}\frac{\sin 2k\theta}{A} \end{aligned} \tag{3.95}$$

The already known effective damping can be also introduced in (3.95), but the effect of the collisions cannot be reduced to the increased damping. The main nonlinear effect here is the infinite number of resonances. A stationary solution to (3.95) can be obtained by setting its right hand sides to zero:

$$\begin{aligned} \frac{\beta_{eff}}{2k}A_* &= -\frac{2\varepsilon}{3\pi\left(4k^2-1\right)}\cos 2k\theta_* \\ \delta A_* + \frac{2\Delta}{\pi} &= \frac{2\varepsilon}{3\pi\left(4k^2-1\right)}\sin 2k\theta_* \end{aligned} \tag{3.96}$$

Eliminating the phase difference θ_* one obtains an equation for the stationary oscillation amplitude:

$$\beta_k{}^2 A_*{}^2 + \left(\delta A_* + \frac{2\Delta}{\pi}\right)^2 = \frac{4\varepsilon_k{}^2}{\pi^2};$$
$$\beta_k = \frac{\frac{1}{2}\beta + \frac{1-R}{\pi}}{2k};\ \varepsilon_k = \frac{\varepsilon}{3\left(4k^2-1\right)} \tag{3.97}$$

This equation has two solutions:

$$A_{1,2} = \frac{2}{\pi}\frac{-\delta\Delta \pm \sqrt{\varepsilon_k{}^2\left(\beta_k{}^2+\delta^2\right) - \Delta^2\beta_k{}^2}}{\beta_k{}^2+\delta^2} \tag{3.98}$$

The expression under the square root must be positive in order for the stationary condition to exist:

$$\Delta \le \varepsilon_k \sqrt{1 - \frac{\delta^2}{\beta_k{}^2}} \tag{3.99}$$

3.5.4 Stability of the Stationary Solutions

Stability of the stationary solutions must be analyzed in order to distinguish between the two solutions (3.98). This can be done quite easily if we rewrite (3.95) as follows:

$$\dot{A} = -2k\beta_k A - \frac{2k\varepsilon_k}{\pi}\cos 2k\theta$$
$$\dot{\theta} = -\delta - \frac{2\Delta}{\pi A} + \frac{\varepsilon_k}{\pi}\frac{\sin 2k\theta}{A} \tag{3.100}$$

Equation in variations can be obtained if we linearize (3.100) in the vicinity of the stationary solution (3.96):

$$A = A_* + \tilde{A};\ \theta = \theta_* + \tilde{\theta}$$
$$\dot{\tilde{A}} = -2k\beta_k\tilde{A} + \frac{2k\varepsilon_k}{\pi}\left(\sin 2k\theta_*\right)2k\tilde{\theta}$$
$$\dot{\tilde{\theta}} = \left(\frac{2\Delta - \varepsilon_k \sin 2k\theta_*}{\pi A_*{}^2}\right)\tilde{A} + \frac{\varepsilon_k}{\pi A_*}\left(\cos 2k\theta_*\right)2k\tilde{\theta} \tag{3.101}$$

The corresponding characteristic equation is given as follows:

$$\det \begin{vmatrix} \lambda + \beta_{eff} & \left(4k^2 \varepsilon_k / \pi\right) \sin \theta_* \\ \delta / A_* & \lambda + \beta_{eff} \end{vmatrix} = 0 \tag{3.102}$$

The relationships (3.96) for the stationary solutions are taken into account. (3.102) is a simple quadratic equation for the natural values. It is necessary and sufficient for the asymptotic stability of the stationary solutions, that the coefficients of the characteristic polynomial are positive (which means that the effective stiffness and effective damping must be positive). These conditions can be written as follows:

$$\begin{aligned} \beta_{eff} &> 0 \\ A_* &> -\frac{2\delta\Delta}{\pi\left(\beta_k{}^2 + \delta^2\right)} \end{aligned} \tag{3.103}$$

Comparing (3.103) with the stationary solutions for the amplitude (3.98) one can see that only the solution with plus in front of the square root is stable.

3.5.5 Discussion of the Results, Comparison between Analytic and Numeric Predictions

The following figure shows the already mentioned countable set of resonances in our system.

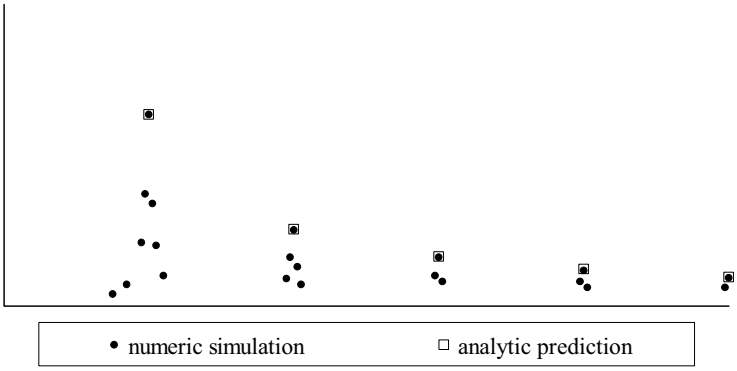

Fig. 3.16. Numeric simulations for the resonance sets and analytic prediction for the maximal amplitude in each resonance set

It is important to notice that the set of resonances is not a consequence of a poly-harmonic excitation but a response of a nonlinear system to a pure harmonic excitation. Figures 3.17 and 3.18 show the zoomed comparison for the main reso-

nance $k=1$ and illustrate the accuracy of the approximate solution. The analysis was performed for the following parameter values:

$\beta=0.01,\ \varepsilon=0.01,\ R=0.95,\ \Delta=0$ for the results shown in Fig. 3.17

$\beta=0.01,\ \varepsilon=0.01,\ R=0.6,\ \Delta=0$ for the results shown in Fig. 3.18.

The accuracy of the approximate analytic prediction remains quite good even in the second case although the collisions are not really "almost elastic" any more. The corresponding "small" parameter $1-R=0.4$ is not small, so one could expect not only quantitative (about 40%) but also qualitative discrepancies. But it is obviously not the case and the prediction (3.98) remains acceptable.

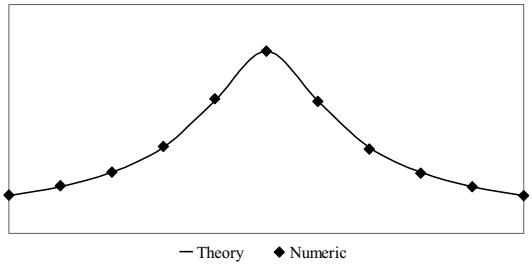

Fig. 3.17. Comparison between analytic and numeric predictions: R=0.95.

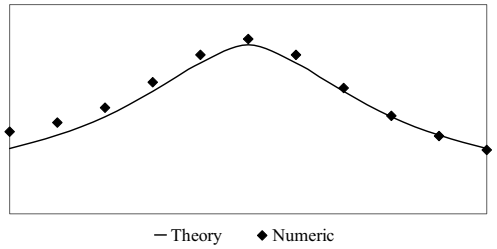

Fig. 3.18. Comparison between analytic and numeric predictions: R=0.6.

Figure 3.19 illustrates how the gap Δ influences the resonance's shape. It is interesting that the maximal amplitude does not depend on Δ, but this constant value corresponds to continuously varying excitations frequency. This result is similar to that in the previous section. Figure 3.20 confirms this statement comparing analytic prediction for maximal oscillation amplitude over all excitation frequencies for a given gap Δ with numerical simulation results. The small discrepancy (the maximal numerical amplitude is not exactly constant but increases slightly with the increasing gap) has the magnitude order of the small parameter and can be obtained if necessary by analyzing the second order approximation like it was done in the previous section. We omit here this analysis in order to avoid unnecessary repetitions.

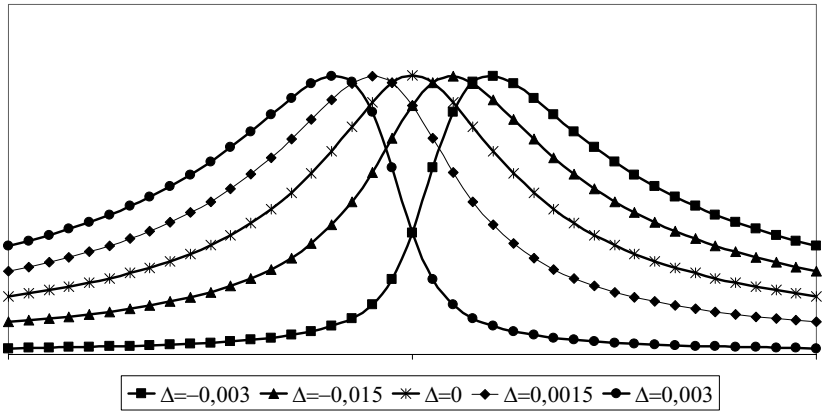

Fig. 3.19. Resonant curves (amplitude vs. frequency) for different gap values. Approximate analytic prediction.

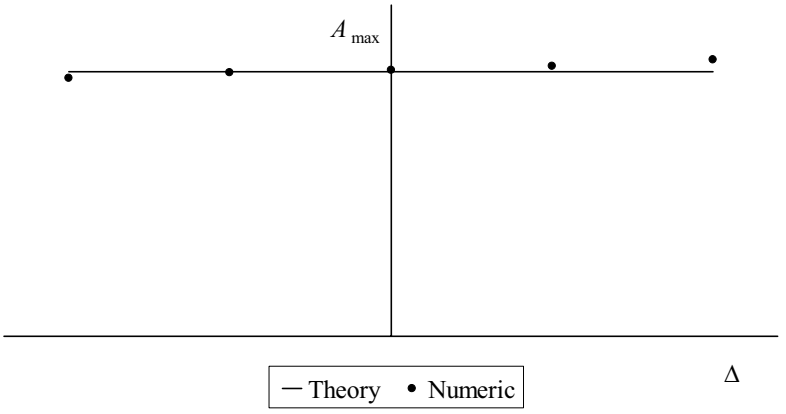

Fig. 3.20. Comparison between analytic and numeric predictions for the maximal resonant amplitude as a function of the gap length

3.6. Nonlinear Resonance of the Externally Excited Oscillator in a Clearance

The last example we are going to discuss in this chapter concerns dynamical properties of the resonantly excited oscillator in a clearance.

3.6.1 Equations of Motion and the Unfolding Transformation

Fig. 3.21 displays the system under consideration: a mass described by its coordinate $s(t)$ attached to a damped linear spring and excited by a harmonic force. The mass is placed in a clearance which length is $2d$ and can collide with each side of it. The static equilibrium point of the mass is in the middle of the clearance.

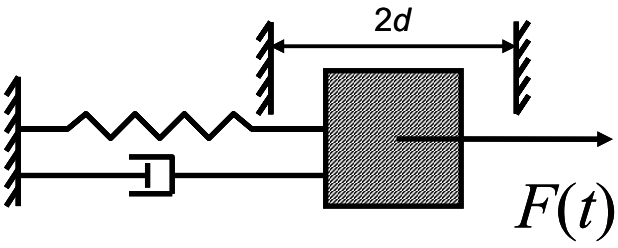

Fig. 3.21. The classical oscillator limited at one side and excited by an external force

The equations of motion and corresponding kinematic conditions describing collisions can be written in an undimensioned form as follows (we consider the case of a harmonic excitation):

$$\begin{aligned} &\ddot{s} + \beta\dot{s} + s = a\sin\omega t, \quad \text{if} \quad |s| < d \\ &s_+ = s_-; \quad \dot{s}_+ = -R\dot{s}_-, \quad \text{if} \quad |s| = d \end{aligned} \tag{3.104}$$

This system according to its formulation seems to differ only insignificantly from the one discussed in the previous section. However, as we already know due to the analysis of the self excited systems, a mass, which motion is restricted at one side, is (in the unfolded form) similar to a linear oscillator. A mass in a clearance in the unfolded form does not even oscillate. Its motion is similar to the rotation of a pendulum in an appropriate periodic field and its behavior is significantly nonlinear.

The unfolding transformation used in the section 3.4 can be applied here too. The corresponding functions are shown in Fig. 3.11

$$\begin{aligned} &s = \frac{2d}{\pi}\Pi(z); \ \Pi(z) = \int_0^z M(\zeta)\,d\zeta = \arcsin(\sin z); \\ &M(z) = \operatorname{sgn}(\cos z) \end{aligned} \tag{3.105}$$

Applying this transformation to the system (3.104) one can easily obtain equations governing the unfolded variable:

$$\ddot{z}+\beta\dot{z}+\Pi(z)M(z)=\varepsilon M(z)\sin\omega t,\quad z\neq\frac{\pi}{2}+\pi n$$
$$\dot{z}_{+}-\dot{z}_{-}=-(1-R)\dot{z}_{-},\quad z=\frac{\pi}{2}+\pi n \tag{3.106}$$
$$\varepsilon=\frac{\pi a}{2d}$$

3.6.2 Analyzing the Unperturbed System and Introducing Slow and Fast Variables

We suppose here as usual in the resonant problems that damping, dissipation due to collisions and amplitude of the excitation force are small. The unperturbed system corresponding to (3.106) is a conservative one. It means, we assume $\beta=0;\ \varepsilon=0;\ R=1$. In this case the full energy of the pendulum is constant.

$$E=\frac{1}{2}\dot{z}_0^{\,2}+Q(z_0);\ Q(z)=\int_0^z M(\zeta)\Pi(\zeta)d\zeta \tag{3.107}$$

Resolving this equation with respect to the pendulum rotation speed (only the rotation case is considered here) we obtain a first order differential equation which can be easily integrated.

$$\dot{z}_0=\sqrt{2(E-Q(z_0))}\quad\Rightarrow\qquad t=t_0+\int_{-\frac{\pi}{2}}^{z_0}\frac{d\zeta}{\sqrt{2(E-Q(\zeta))}} \tag{3.108}$$

Function $t(z_0)$ calculated according to (3.108) is displayed in Fig. 3.22. It is important to notice that the full time between two collisions corresponding to the period of motion with respect to the variable z_0 i.e. $-\frac{\pi}{2}<z_0<\frac{\pi}{2}$ can be found quite easily (compare section 3.4):

$$\Delta T=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\zeta}{\sqrt{2(E-Q(\zeta))}}=2\arcsin\left(\frac{\pi}{2\sqrt{2E}}\right) \tag{3.109}$$

The solution (3.107), (3.108) can be considered as the basis for the analysis of the full system (3.105).

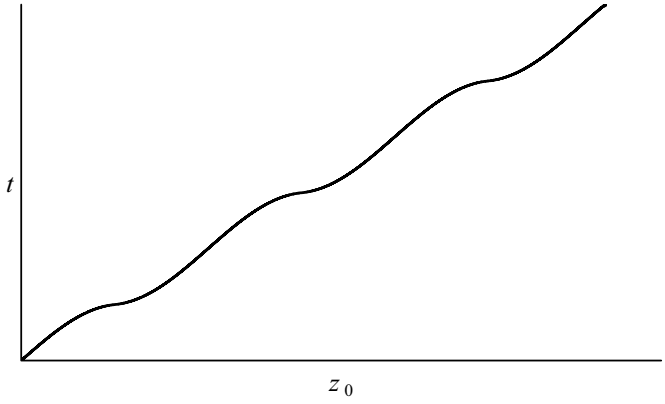

Fig. 3.22. Solution of the unperturbed system

We introduce the energy as the new unknown function instead of the pendulum's speed $\dot{z}$, replace the time by a new uniformly rotating phase and convert to z as the new independent variable:

$$\psi = \omega t \tag{3.110}$$

$$\begin{aligned} E' &= -\beta\sqrt{2\left(E-Q(z)\right)} + \varepsilon M(z)\sin\psi, \;\; z \neq \frac{\pi}{2} + \pi n \\ E_+ - E_- &= -\left(1-R^2\right)\left(E - \frac{\pi^2}{8}\right), \;\; z = \frac{\pi}{2} + \pi n \\ \psi' &= \frac{\omega}{\sqrt{2\left(E-Q(z)\right)}} \\ z' &= 1 \end{aligned} \tag{3.111}$$

System (3.111) is very interesting. It contains one slow variable E and two fast rotation phases (i.e. monotonously increasing variables) ψ and z. Similar to the previous section we are interested in the resonant motions. But the difference between this system and that in the paragraph 3.5 is the fact that ψ' is not constant here. It depends both on fast and slow variables. What is the resonance in this case?

3.6.3 Resonances in the Significantly Nonlinear System

The only difference in the resonance definition between the almost linear case considered above and significantly nonlinear case here is as follows. In the almost linear case the resonant surface was determined in the parameter space. In the nonlinear case it can be determined as *a subset of the phase space corresponding to the set of the slow variables' values* (which are actually constants in the unperturbed equations) *for which the time averages of the right hand sides of the full equations become discontinuous*. The resonant solutions are solutions of our system for slow variables being in the vicinity of the resonant surface (in sense of the small parameter). We are going to illustrate this definition considering the system (3.111). The corresponding unperturbed equations are quite easy:

$$\left.\begin{aligned} E' &= 0 \\ \psi' &= \frac{\omega}{\sqrt{2\left(E-Q(z)\right)}} \\ z' &= 1 \end{aligned}\right\} \Rightarrow \psi = \psi_0 + \omega \int\limits_{-\frac{\pi}{2}}^{z} \frac{d\zeta}{\sqrt{2\left(E-Q(\zeta)\right)}} \tag{3.112}$$

Now we have to average the right hand sides of (3.111) along the solution (3.112) and find out for which values of the energy this average becomes discontinuous. The only term which can become discontinuous is

$$\left\langle M(z)\sin\left(\omega \int\limits_{-\frac{\pi}{2}}^{z} \frac{d\zeta}{\sqrt{2\left(E-Q(\zeta)\right)}} + \psi_0\right)\right\rangle \tag{3.113}$$

It is easy to notice considering Fig. 3.22 and the relationship (3.109) that the function $M(z)$ is 2π -periodic, in other words its frequency is 1. The average (3.113) is not equal to zero if the argument of the sin-function has the same period as the function $M(z)$. It means the main resonant surface corresponds to the energy level fulfilling the condition

$$\omega\Delta T = \pi \Leftrightarrow \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\zeta}{\sqrt{2\left(E-Q(\zeta)\right)}} = \frac{\pi}{\omega} \tag{3.114}$$

This is not the only resonant surface in our problem. A countable number of resonant surfaces exist in this system, similar to the previous section. They fulfill the relationship

$$\omega\Delta T = (2k+1)\pi;\ \ k = 1,2,3,\ldots \tag{3.115}$$

However we restrict our analysis to the main case (3.114) in order to avoid unnecessary complications. It means we investigate oscillations of the same frequency as the external excitation. The equation (3.114) can be resolved with respect to the resonant energy level:

$$\omega = \frac{\pi}{2\arcsin\left(\dfrac{\pi}{2\sqrt{2E_R}}\right)} \Leftrightarrow E_R = \frac{\pi^2}{8\sin^2\left(\dfrac{\pi}{2\omega}\right)} \tag{3.116}$$

The energy must be sufficient large according to the sense of our unfolding transformation:

$$E > \pi^2/8 \tag{3.117}$$

This inequality is equivalent to the corresponding relationship for the excitation frequency:

$$\omega > 1 \tag{3.118}$$

3.6.4 Averaging in the Vicinity of the Main Nonlinear Resonance

Now we would like to analyze solutions to the equations (3.87) in the vicinity of the main resonant surface (3.116). The system (3.111) must be transformed to the "standard form". It can be done if we investigate its solutions in the $\sqrt{\varepsilon}$-vicinity of the resonance and consider $\sqrt{\varepsilon}$ as the new small parameter. The details are discussed in the Chapter 5 which is mainly devoted to systems interacting with an energy source of limited power (induction motor). It is easy to notice that an intention to analyze the system in the ε-vicinity of the resonance would not lead to a system with slow variables.

We introduce a new variable describing the difference between the actual energy and the resonant energy level:

$$E = E_R + \sqrt{\varepsilon}u \tag{3.119}$$

Then the equations of motion can be rewritten as follows if we neglect all the terms which are smaller than ε, i.e. the terms $o\left(\sqrt{\varepsilon}\right)$:

$$
\begin{aligned}
&u' = -\beta\varepsilon^{-\frac{1}{2}}\sqrt{2\left(E_R - Q(z)\right)} + \varepsilon^{\frac{1}{2}} M(z)\sin\psi,\ z \neq \tfrac{\pi}{2} + \pi n \\
&u_+ - u_- = -\left(1 - R^2\right)\varepsilon^{-\frac{1}{2}}\left(E_R - \tfrac{\pi^2}{8}\right),\quad z = \tfrac{\pi}{2} + \pi n \\
&\psi' = \frac{\omega}{\sqrt{2\left(E_R - Q(z)\right)}} - \sqrt{\varepsilon} u\left[2\left(E_R - Q(z)\right)\right]^{-\frac{3}{2}}
\end{aligned}
\tag{3.120}
$$

Variable u is slow with respect to the parameter $\sqrt{\varepsilon}$ because

$$
\begin{aligned}
&\beta = O(\varepsilon),\ 1 - R = O(\varepsilon) \\
&\Rightarrow \beta\varepsilon^{-\frac{1}{2}} = O\left(\varepsilon^{\frac{1}{2}}\right),\ (1 - R)\varepsilon^{-\frac{1}{2}} = O\left(\varepsilon^{\frac{1}{2}}\right)
\end{aligned}
\tag{3.121}
$$

But the variable ψ contains a fast term. In order to eliminate it the following transformation can be performed:

$$
\psi = \omega R(z) + \theta;\ R(z) = \int_{-\frac{\pi}{2}}^{z} \frac{dz}{\sqrt{2\left(E_R - Q(z)\right)}} \tag{3.122}
$$

Finally the equations in the "standard form" for systems with almost elastic collisions can be obtained:

$$
\begin{aligned}
u' = &-\frac{\beta}{\sqrt{\varepsilon}}\sqrt{2\left(E_R - Q(z)\right)} \\
&+\sqrt{\varepsilon} M(z)\sin\left(\omega R(z) + \theta\right),\ z \neq \tfrac{\pi}{2} + \pi n \\
u_+ - u_- = &-\frac{1 - R^2}{\sqrt{\varepsilon}}\left(E_R - \tfrac{\pi^2}{8}\right),\quad z = \tfrac{\pi}{2} + \pi n \\
\theta' = &-\sqrt{\varepsilon}\frac{u}{\left[2\left(E_R - Q(z)\right)\right]^{\frac{3}{2}}}
\end{aligned}
\tag{3.123}
$$

System (3.123) is suitable for averaging. We need to calculate the following averages:

$$\left\langle \sqrt{2\left(E_R - Q(z)\right)} \right\rangle = \frac{1}{2}\sqrt{2E_R - \frac{\pi^2}{4}} + \frac{2}{\pi}E_R \arcsin\left(\frac{\pi}{2\sqrt{2E_R}}\right)$$

$$\left\langle \frac{1}{\left[2\left(E_R - Q(z)\right)\right]^{\frac{3}{2}}} \right\rangle = \frac{1}{2E_R\sqrt{2E_R - \frac{\pi^2}{4}}} \tag{3.124}$$

The averages containing $R(z)$ can be also calculated easily.

$$\left\langle M(z)\sin\left(\omega R(z)\right)\right\rangle = 0$$

$$\left\langle M(z)\cos\left(\omega R(z)\right)\right\rangle = \frac{\sqrt{2E_R}}{2\omega} \tag{3.125}$$

The first order approximation to (3.123) can be written as follows:

$$\theta' = -\frac{\sqrt{\varepsilon}u}{2E_R\sqrt{2E_R - \frac{\pi^2}{4}}}$$

$$u' = -\frac{\beta}{\sqrt{\varepsilon}}\left[\sqrt{E_R - \frac{\pi^2}{8}} + \frac{2}{\pi}\arcsin\left(\frac{\pi}{2\sqrt{2E_R}}\right)\right] \tag{3.126}$$

$$-\frac{1-R^2}{\pi\sqrt{\varepsilon}}\left(E_R - \frac{\pi^2}{8}\right) + \sqrt{\varepsilon}\frac{\sqrt{2E_R}}{2\omega}\sin\theta$$

3.6.5 Equations Governing the Slow Motions; Discussion of the Results

The system (3.126) is actually very simple. It can be easily seen if we eliminate the variable u and write one second order equation for the phase difference between the excitation and the rotation of the unfolded pendulum θ:

$$\theta'' + \varepsilon k^2 \sin\theta = M_0$$

$$k^2 = \frac{\sqrt{2E_R}}{2\omega\sqrt{2E_R}\sqrt{2E_R - \frac{\pi^2}{4}}}$$

$$M_0 = \frac{1-R^2}{\pi}\frac{\sqrt{E_R - \frac{\pi^2}{8}}}{2\sqrt{2}E_R} + \beta\frac{\sqrt{E_R - \frac{\pi^2}{8}} + \frac{2}{\pi}\arcsin\left(\frac{\pi}{2\sqrt{2E_R}}\right)}{2\sqrt{2}E_R\sqrt{E_R - \frac{\pi^2}{8}}} \tag{3.127}$$

This equation describes motions of a pendulum with an additional constant external torque. This result is very typical for the resonant problems and we will meet similar equations many times. The main properties of this system are obvious. An equilibrium point (corresponding to stationary resonant oscillations) exists if the external torque is smaller than the "stiffness" of the pendulum. Two solutions of the equation (3.127) exist if this condition is fulfilled. One of them corresponding to positive $\cos\theta$ is stable and another one is unstable. The existence condition for the stationary regime can be considered as a requirement to the minimal excitation amplitude (the relationship (3.116) must be taken into account).

$$M_0 < \varepsilon k^2 \Leftrightarrow$$

$$\varepsilon > \frac{4\beta}{\pi}\sin\left(\frac{\pi}{2\omega}\right) + \sqrt{2}\beta\omega\cos\left(\frac{\pi}{2\omega}\right) + \frac{1-R^2}{2}\omega\frac{\cos^2\left(\frac{\pi}{2\omega}\right)}{\sin\left(\frac{\pi}{2\omega}\right)} \tag{3.128}$$

The phase portrait of the equivalent resonant pendulum is displayed in Fig. 3.23.

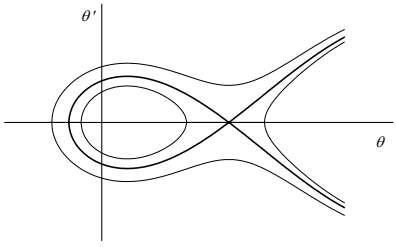

Fig. 3.23. The phase portrait of the equivalent resonant pendulum; the homoclinic loop (thick line) limits the attraction area of the stationary solution (according to the first order approximation)

The minimal excitation level according to (3.128) is shown in Fig. 3.24.

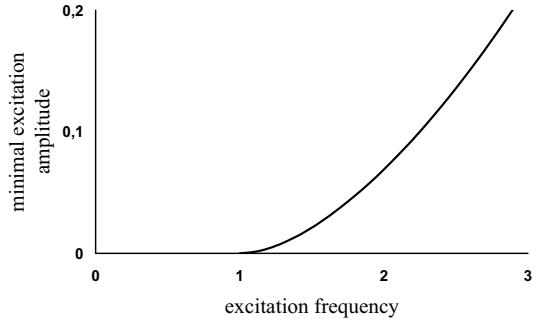

Fig. 3.24. The minimal excitation which is necessary for the resonance to exist

3.6.6 Comparison between Analytic and Numeric predictions

Figure 3.25 shows a comparison between the analytically predicted oscillation frequency in the resonant area and numerical simulation results.

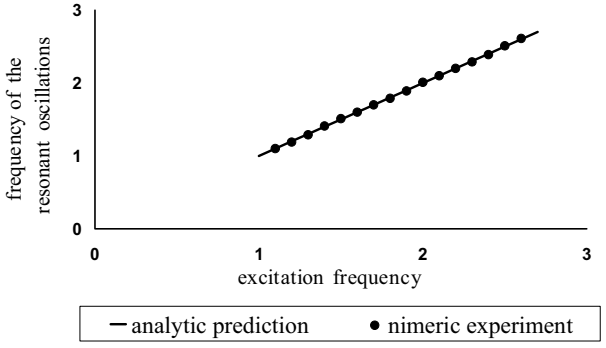

Fig. 3.25. Comparison between analytic and numeric predictions

The results are very accurate even though the formal inaccuracy of the analytic prediction has the magnitude order $\sqrt{\varepsilon}$.

The higher order resonances also exist in this system as it was already mentioned. The corresponding analytic and numeric results are shown in Fig. 3.26.

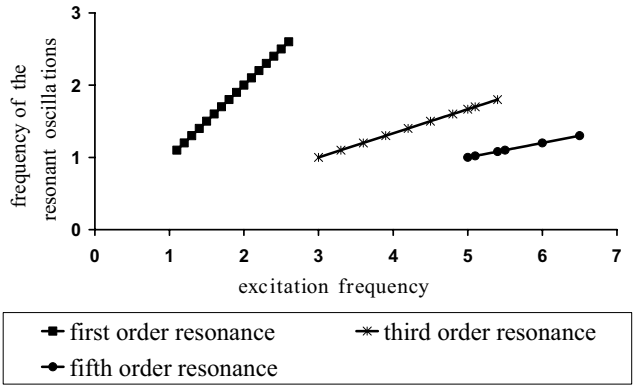

Fig. 3.26. First and higher order resonances of the mass in the clearance

Comparing this figure with the results in the previous section one can notice that significantly nonlinear resonance considered here exists in the wider frequency range and that several different resonant regimes can exist for the same excitation parameters depending on the initial conditions. It means the question about the attraction area for each regime becomes sensible. This question for similar problems is discussed in Chapter 5.

3.7 Conclusions

Oscillations of systems with collisions are significantly nonlinear. An efficient approach for analyzing these oscillations is based on the unfolding transformations. It leads to systems with small discontinuities of the velocities if the collisions in the original system were absolutely or almost elastic.

Systems limited at one side are in the unfolded form similar to a linear oscillator. Systems limited from both sides are significantly nonlinear. This statement is illustrated by two groups of examples.

The classical system with the “mass-on-moving-belt” is considered as the first example. It is necessary to take the friction’s nonlinearity into account in order to obtain a stable limit cycle in the system limited from one side. This behavior is qualitatively similar to that of a not limited oscillator. The behavior of the same system in a clearance is different. A stable limit cycle is possible without any additional assumptions concerning the nonlinearity of friction. Collisions are able to stabilize the limit cycle themselves. The system can be interpreted as a frequency transformer. It is possible to control the oscillation frequency varying the belt velocity and as a result changing the negative slope of the friction characteristics. The same effect can be achieved if it is possible to change the clearance length. These properties of the mass on moving belt in a clearance open an opportunity to use it or similar systems as a dynamic damper which frequency can be easily ad-

justed in a wide range. It can be also used in order to excite vibrations in machines if the oscillations' frequency must be changed while operating.

The classical externally excited oscillator limited from one or from both sides is considered as the second example. A countable number of sharp resonances are possible in the system limited from one side. Each of them is quite similar to those usual in linear systems.

The problem of the resonant oscillations of the mass in a clearance is much more complex. It can be reduced to the analysis of the rotation of the harmonically excited pendulum in a periodical non smooth potential field. The high energy resonances are investigated. Accurate analytic predictions for the energies of stationary resonant regimes are obtained. It is shown that an infinite number of high energy resonances exist in this system. Investigations concerning the attraction area for particular regimes may be of considerable interest for applications.

4. Systems with Strong Dissipation Due to High Damping or Inelastic Collisions

Systems with small damping, i.e. systems which are close to conservative ones in the unperturbed case, were discussed in the previous section. Now we are going to analyze another type of discontinuous systems.

In many practical cases typically connected with bulk material processing, the collisions cannot be considered as absolutely or almost elastic. If two rigid bodies collide through a layer of a bulk material, the energy dissipation is high. Such a situation is usual in jaw crashers, grinding mills and other machines for mechanic destruction of ore and stones. The same situation is typical for vibrating screens and conveyors, if we are not interested in the motion of each particle, but describing the layer as a whole. The hypothesis of the absolutely inelastic collisions is much more useful in these cases (*cf.* [18, 20, 65]).

These machines can often be described as a combination of an almost conservative part and of a subsystem with strong dissipation. There are some peculiarities in the analysis of strongly damped systems in general and in the description of the systems with inelastic collisions in particular.

First of all, systems with inelastic collisions cannot be described by systems of differential equations which are valid for all times. The continuous differential equations are valid between the collisions and state variables after each collision can be calculated as functions of state variables before the collision. There are two different types of inelastic collisions. The first one corresponds to collisions with a finite time interval of contact between the colliding bodies. In these cases systems before and during the collision have different orders. The main interest in this chapter is paid to this type of motion. The second type corresponds to inelastic collisions with infinitesimally short contacts. This type of motion is less important from the practical point of view and is not analyzed here. The corresponding examples can be found in [15, 18, 58].

The dissipative subsystem moves in many important cases as a slave of the almost conservative subsystem (at least in the first order approximation). This important property can be used in order to develop efficient analytical methods. This approach allows reducing the order of the system under consideration significantly.

The last statement is quite general and is not restricted to the systems with inelastic collisions. In the paragraph 4.1 the corresponding asymptotic procedure is developed for systems with strong linear damping. The approach is illustrated by an elementary linear example in the section 4.2. Another much more complex ex-

ample describing the unusual effect of the high-frequency excitation on the strongly damped control system can be found in Chapter 6.

The same basic idea enables us to develop the variable order discontinuous averaging procedure, which is useful for analyzing inelastic collision oscillators (section 4.3). Compared to the oscillator with almost elastic collision the analysis must be restricted to the given collisions sequence, for example to the given number of collisions for each oscillation period. This restriction is constraining from the theoretical point of view, but it is not so critical in practice, because mainly dynamic regimes with the most intensive collisions are desired for bulk material processing. The corresponding regime of collisions is usually evident, and the analysis can be concentrated upon it.

The classical double restricted impact oscillator with inelastic collisions is considered in section 4.4. The aim is to demonstrate the main ideas and methods using a simple and physically clear object. A similar approach is illustrated in section 4.5 by a mass restricted on one side and oscillating in the gravity field. This model is typical for vibrating screens and conveyors. The described methods are applied for a more complex and realistic model of resonant crasher excited by an inertial source of limited power in the Chapter 5, dealing with the significantly nonlinear resonance.

4.1. Averaging in Systems with Strong Linear Damping

4.1.1. The Basic Idea

Let us start with the following scalar equation illustrating the basic properties of systems with the strong linear damping:

$$\begin{aligned} x' &= -kx + \varepsilon X\left(x,t\right) \\ x\left(0\right) &= x_0; k > 0;\ k = O\left(1\right) \end{aligned} \tag{4.1}$$

Here $X\left(x,t+2\pi\right) = X\left(x,t\right)$ is a bounded continuous periodic function: $\left\|X\left(x,t\right)\right\| \le M_X = O\left(1\right)$.

It is not possible to transform this equation to the standard form for averaging, because the unperturbed equation is

$$x_u' = -kx_u \tag{4.2}$$

Its general solution is

$$x_u = Ae^{-kt} \tag{4.3}$$

Considering A as the new unknown function in (4.1) one obtains the following equation:

$$A' = \varepsilon e^{kt} X\left(Ae^{-kt}, t\right) \tag{4.4}$$

It seems to be an equation in standard form, but it is actually not, because its right hand side is not bounded as a function of t. So the standard averaging cannot be applied to (4.4). On the other hand it is obvious that general solutions to (4.1) and to (4.2) are in some sense close to each other.

Let us consider the following initial value problem alongside the equations (4.1):

$$\xi_1' = -k\xi_1;\ \xi_1(0) = x_0 \tag{4.5}$$

In order to compare solutions to (4.1) and to (4.5) let us rewrite (4.1) as an integral equation:

$$x = x_0 e^{-kt} + \varepsilon e^{-kt} \int_0^t e^{k\tau} X(x, \tau)\, d\tau \tag{4.6}$$

Supposing x to be a solution to (4.6) and substituting it into (4.1) it is easy to see that it is also a solution to (4.1) satisfying the right initial condition. Thus according to the uniqueness of solution to the initial value problem (4.1), we can conclude that (4.6) is equivalent to (4.1). On the other hand the solution to (4.5) corresponding to the same initial condition is:

$$\xi_1 = x_0 e^{-kt} \tag{4.7}$$

Comparing (4.6) with (4.7) we can estimate the difference:

$$\begin{aligned} \left\|x - \xi_1\right\| &= \left\| \varepsilon e^{-kt} \int_0^t e^{k\tau} X(x, \tau)\, d\tau \right\| \\ &\le \varepsilon e^{-kt} \int_0^t e^{k\tau} \left\| X(x, \tau) \right\| d\tau \le \varepsilon e^{-kt} \int_0^t e^{k\tau} M_X\, d\tau \\ &= \varepsilon e^{-kt} \frac{M_X}{k}\left(e^{k\tau} - 1\right) = \varepsilon \frac{M_X}{k}\left(1 - e^{-k\tau}\right) \le \varepsilon \frac{M_X}{k} \end{aligned} \tag{4.8}$$

This result is almost trivial. Influence of the initial condition decreases according to the linear part of the equation and the perturbation is small. So the stationary solution also must be of the same order as the perturbation itself.

This idea can be developed further if we consider the second approximation. How fast the solution to (4.1) will reach the $O(\varepsilon)$-vicinity of zero?

If $x_0 = O(1)$ then

$$\begin{aligned} &\xi_1 = O(\varepsilon) \Leftrightarrow e^{-kt} = O(\varepsilon) \Leftrightarrow kt = \ln O(1/\varepsilon) \\ &\Rightarrow t = O\left(\ln(1/\varepsilon)\right) \end{aligned} \tag{4.9}$$

Let us investigate solutions for sufficient large time values, i.e.

$$t_* \geq O\left(\ln(1/\varepsilon)\right) \tag{4.10}$$

In particular we can get the following estimate

$$\|\xi_1\| \leq \varepsilon \frac{M_X}{k} \text{ for } t \geq \frac{1}{k \ln\left(\dfrac{k\|x_0\|}{\varepsilon M_X}\right)} \tag{4.11}$$

It means that for the sufficient large time (corresponding to (4.11)) we can estimate the solution as follows:

$$\|x\| \leq \varepsilon \frac{2M_X}{k} \tag{4.12}$$

Now it is easy to find the second order approximation to (4.1) if we suppose X to be Lipschitz-continuous function in the vicinity of $x = 0$:

$$\|X(x_1,t) - X(x_2,t)\| \leq L_X \|x_1 - x_2\| \tag{4.13}$$

Equation (4.1) can be rewritten as follows:

$$\begin{aligned} &x' = -kx + \varepsilon X(x,t) = -kx + \varepsilon X(0,t) + \varepsilon^2 G(x,t) \\ &x(0) = x_0 \end{aligned} \tag{4.14}$$

Function G is bounded for the time interval (4.11):

$$\|G(x,t)\| = \frac{1}{\varepsilon}\|X(x,t) - X(0,t)\| \leq L_X \frac{\|x\|}{\varepsilon} = \frac{2L_X M_X}{k} \tag{4.15}$$

We can rewrite (4.14) in the integral form similarly to (4.6):

$$x = x_0 e^{-kt} + \varepsilon e^{-kt} \int_0^t e^{k\tau} X(0,\tau)\, d\tau + \varepsilon^2 e^{-kt} \int_0^t e^{k\tau} G(x,\tau)\, d\tau \tag{4.16}$$

Let us compare its solution to the corresponding approximate equation

$$\begin{aligned}&\xi_2' = -k\xi_2 + \varepsilon X(0,t);\ \ \xi_2(0) = x_0 \Leftrightarrow \\ &\xi_2 = x_0 e^{-kt} + \varepsilon e^{-kt}\int_0^t e^{k\tau} X(0,\tau)\,d\tau\end{aligned} \tag{4.17}$$

The analysis does not differ in any relation from (4.8) and the result is:

$$\|x - \xi_2\| = \left\|\varepsilon^2 e^{-kt}\int_0^t e^{k\tau} G(x,\tau)\,d\tau\right\| \le \varepsilon^2 \frac{2L_X M_X}{k^2} \tag{4.18}$$

Summarizing one can say that solutions to (4.1) and to (4.17) are close to each other with an error estimated through (4.18) for sufficiently large time (4.11).

This result has nothing to do with averaging. It is obtained only due to the exponential properties of the solutions to (4.1).

4.1.2. Averaging in Systems with Strong Damping with Respect to one or Several Variables

These ideas can be directly combined with the averaging if we consider a system with some additional oscillating degrees of freedom:

$$\begin{aligned}&x' = \varepsilon X(x,y,t)\\ &y' = -ky + \varepsilon Y(x,y,t)\\ &x(0) = x_0; y(0) = y_0\end{aligned} \tag{4.19}$$

Functions X and Y are assumed to be periodic with respect to t and bounded and Lipschitz-continuous with respect to x and y together with their first derivatives.

We are going to build the corresponding approximate system combining the averaging ideas with the results of the just performed analysis. The "averaged" approximation to (4.19) has the following form:

$$\begin{aligned}&\xi' = \varepsilon\Xi_1(\xi) + \varepsilon^2\Xi_2(\xi)\\ &\eta' = -k\eta + \varepsilon Y(\xi,0,t)\\ &\xi(0) = x_0; \eta(0) = y_0\end{aligned} \tag{4.20}$$

Functions Ξ_1 and Ξ_2 will be determined later. Solutions to (4.19) and (4.20) must be asymptotically close to each other with an error $O(\varepsilon^2)$ for the time interval

$$O\left(\ln\left(1/\varepsilon\right)\right) < t < O\left(1/\varepsilon\right) \tag{4.21}$$

The first approximation is obvious. According to (4.11), (4.12) $y = O\left(\varepsilon\right)$ for the time interval (4.21). So the equation for x can be rewritten as follows:

$$\begin{aligned} x' &= \varepsilon X\left(x,0,t\right) + O\left(\varepsilon^2\right) \\ y &= O\left(\varepsilon\right) \end{aligned} \tag{4.22}$$

To this system we can apply the standard averaging and obtain

$$\begin{aligned} \xi_1' &= \varepsilon \Xi_1\left(\xi_1\right); \Xi_1\left(\xi_1\right) = \left\langle X\left(\xi_1,0,t\right)\right\rangle \\ \eta_1' &= -k\eta_1 \end{aligned} \tag{4.23}$$

It is clear from (4.23) that the equations for ξ and η in the first approximation are totally separated. The difference between x and ξ_1 is a small oscillating function as it is usual in the averaging:

$$x = \xi_1 + \varepsilon u\left(\xi_1, t\right) \tag{4.24}$$

Taking (4.14) into account we can rewrite the last equation in (4.23) as follows:

$$y' = -ky + \varepsilon Y\left(x,y,t\right) = -ky + \varepsilon Y\left(x,0,t\right) + \varepsilon^2 G\left(x,y,t\right) \tag{4.25}$$

Function G is introduced here similarly to (4.14). We know already that the solution of this equation is close to the solution of the corresponding shortened equation in the relevant time interval

$$\begin{aligned} \eta_2' &= -k\eta_2 + \varepsilon Y\left(x,0,t\right) + O\left(\varepsilon^2\right) \\ \eta_{2\infty} &= \frac{\varepsilon}{k} Y\left(x,0,t\right) + O\left(\varepsilon^2\right) \end{aligned} \tag{4.26}$$

This expression can now be inserted into equation for x:

$$x' = \varepsilon X\left(x, \frac{\varepsilon}{k} Y + O\left(\varepsilon^2\right), t\right) = \varepsilon X\left(x,0,t\right) + \frac{\varepsilon^2}{k}\frac{\partial X}{\partial y} Y + O\left(\varepsilon^3\right) \tag{4.27}$$

Now the standard averaging can be performed as an almost identical transformation:

$$x = \xi + \varepsilon u_1\left(\xi,t\right) + \varepsilon^2 u_2\left(\xi,t\right) + \ldots \tag{4.28}$$

Here ξ is governed by the first equation (4.23). Substituting (4.28) into (4.27), balancing terms of the same ε-order and eliminating singular terms one obtains (4.23) with the following expressions for the functions on the right hand sides:

$$\begin{aligned}
&\xi' = \varepsilon\Xi_1(\xi) + \varepsilon^2\Xi_2(\xi) + O(\varepsilon^2)\\
&\eta' = -k\eta + \varepsilon Y(\xi,0,t) + O(\varepsilon^2)\\
&\xi(0) = x_0; \eta(0) = y_0\\
&\Xi_1(\xi) = \langle X(\xi,0,t)\rangle\\
&u_1 = \int_0^t [X(\xi,0,\tau) - \Xi_1(\xi)]\,d\tau\\
&\Xi_2(\xi) = \left\langle \left.\frac{\partial X}{\partial x}\right|_{y=0} u_1 + \frac{1}{k}\left.\frac{\partial X}{\partial y}\right|_{y=0} Y(\xi,0,t)\right\rangle
\end{aligned} \tag{4.29}$$

The proof validating the shortened form (4.19) is a direct combination of the standard averaging proof with the estimation (4.18).

These results can be easily generalized for the case when y is a vector and $(-k)$ is a matrix which natural values have only negative real parts.

The performed analysis is quite similar to that published by Volosov V. M. and Morgunov B. I. [133]. The "slave" variables y are related to non critical fast variables introduced there.

Looking at (4.26) one can easily notice why we call these variables "slaves". They don't have their own dynamics and can be expressed as functions of the "master" variables x. In other words, if we know the master variables as functions of time, we can calculate the slave variables without solving any additional differential equation.

The next paragraph illustrates this approach.

4.2. Linear Resonance in a Strongly Damped System with Two Degrees of Freedom

4.2.1. Equations of Motion

An elementary mechanical system with two degrees of freedom is analyzed in this section in order to illustrate the averaging procedure for strongly damped systems.

Consider a system shown in Fig. 4.1. It consists of a large mass M to which a small mass m is attached. The relative motion between these two masses is strongly damped (damping coefficient is b). C and c are the spring stiffnesses. The large mass is excited kinematically.

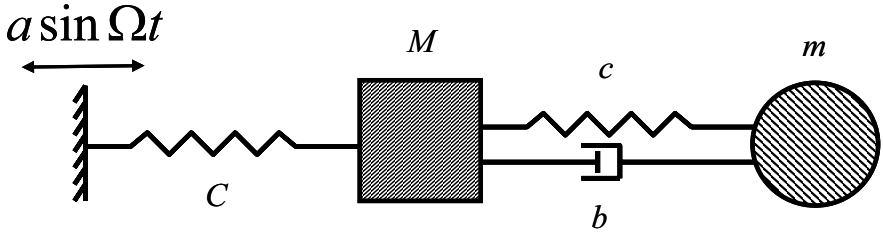

Fig. 4.1. The linear system with two degrees of freedom and strong damping

This is a linear system and its general solution can be found in any textbook on linear oscillations. Nevertheless we analyze it because it demonstrates some important properties of strongly damped systems which are also important for systems with inelastic collisions.

The motions of this system are governed by the following equations:

$$\begin{aligned} M\ddot{x}_1 + Cx_1 + b\left(\dot{x}_1 - \dot{x}_2\right) + c\left(x_1 - x_2\right) &= Ca\sin\Omega t \\ m\ddot{x}_2 + b\left(\dot{x}_2 - \dot{x}_1\right) + c\left(x_2 - x_1\right) &= 0 \end{aligned} \tag{4.30}$$

It is sensible to introduce new variables describing the position of the system's mass center and the relative motion between the masses:

$$s = \frac{Mx_1 + mx_2}{M+m};\ r = x_2 - x_1 \tag{4.31}$$

Equations governing these variables can be written down in the undimensioned form as follows:

$$\begin{aligned} s'' + (1-\mu)(s - \mu r) &= a(1-\mu)\sin\omega t \\ r'' + \frac{2\beta p}{1-\mu}r' + \frac{p^2}{1-\mu}r - (s - \mu r) &= -a\sin\omega t \end{aligned} \tag{4.32}$$

The following variables and parameters are used in these equations:

$$\mu = \frac{m}{M+m};\ k = \sqrt{\frac{C}{M}};\ \lambda = \sqrt{\frac{c}{m}};\ \beta = \frac{b}{2\lambda m};\ \omega = \frac{\Omega}{k};\ p = \frac{\lambda}{k}$$
$$\tau = kt;\ (\ )' = d(\ )/d\tau \tag{4.33}$$

We are going to analyze this system supposing the second mass to be small in comparison to the whole mass of the system. We suppose further the excitation's amplitude to be small compared to oscillation's amplitude, i.e.

$$\mu << 1;\ a << 1 \tag{4.34}$$

However we do not assume the damping to be small, i.e. $\beta = O(1)$.

Now we can linearize the equations (4.32) with respect to the introduced small parameters:

$$\begin{aligned} &s'' + s = \mu s + \mu r + a\sin\omega t \\ &r'' + 2\beta p r' + p^2 r - s = -2\mu\beta p r' - \mu\left(p^2+1\right)r - a\sin\omega t \end{aligned} \tag{4.35}$$

This is the system we are going to analyze.

4.2.2. Perturbation Analysis. Transformation to the Form suitable for Averaging

Consider the unperturbed system related to (4.35):

$$\begin{aligned} &s_0'' + s_0 = 0 \\ &r_0'' + 2\beta p r_0' + p^2 r_0 - s_0 = 0 \end{aligned} \tag{4.36}$$

One can easily notice that the subsystem governing the variable s is in this approximation separated from the subsystem governing the variable r. The variable s_0 is governed by a conservative equation (there is no damping in the corresponding unperturbed subsystem). The variable r_0 is to the contrary strongly damped and the solution s_0 is an external excitation for it. In other words, we can write down the general solution for s_0:

$$s_0 = A\sin\varphi;\ s_0' = A\cos\varphi;\ A = const; \varphi = \tau + const \tag{4.37}$$

Then we can rewrite the equation for r_0:

$$r_0'' + 2\beta p r_0' + p^2 r_0 = A\sin\varphi \tag{4.38}$$

Our objective is to transform the equations (4.35) to the form (4.19) which contains the damping terms explicitly. In order to achieve this aim we do not need the general solution to (4.38). We need only the particular solution corresponding to the enforced oscillations. It can be found in the following form:

$$r_0 = q_0 + \lambda_1 A\sin\varphi + \lambda_2 A\cos\varphi;\ \ q_0 = const$$
$$\lambda_1 = \frac{p^2-1}{\left(p^2-1\right)^2+4p^2\beta^2};\ \lambda_2 = -\frac{2p\beta}{\left(p^2-1\right)^2+4p^2\beta^2} \tag{4.39}$$

The equation governing the variable q is:

$$q_0'' + 2\beta p q_0'' + p^2 q_0 = 0 \tag{4.40}$$

It can be easily transformed to a set of simple first order differential equations if we apply the following transformation:

$$q_0 = u_0\sin\psi_0;\ q_0' = -\beta p u_0\sin\psi_0 + p\sqrt{1-\beta^2}u_0\cos\psi_0 \tag{4.41}$$

This transformation corresponds to the general solution to (4.40):

$$q_0 = const\cdot e^{-\beta pt}\sin\left(p\sqrt{1-\beta^2}t + const\right) \tag{4.42}$$

Variables u and ψ in this approximation are governed by the following equations:

$$u_0' = -\beta p u_0$$
$$\psi_0' = p\sqrt{1-\beta^2} \tag{4.43}$$

Let us consider A,φ and q as the new variables and apply (4.37) and (4.39) as the variable transformation to the equations (4.35):

$$s = A\sin\varphi;\ s' = A\cos\varphi$$
$$r = q + \lambda_1 A\sin\varphi + \lambda_2 A\cos\varphi \tag{4.44}$$

The result is the system governing the new variables:

$$A' = \mu A\left(\frac{q}{A} + \left(1+\lambda_1\right)\sin\varphi + \lambda_2\cos\varphi\right)\cos\varphi + a\sin\omega t\cos\varphi$$
$$\varphi' = 1 - \mu\left(\frac{q}{A} + \left(1+\lambda_1\right)\sin\varphi + \lambda_2\cos\varphi\right)\sin\varphi - \frac{a}{A}\sin\omega t\sin\varphi \tag{4.45}$$
$$q'' + 2\beta p q' + p^2 q = -2\mu\beta p\left(q' + \lambda_1 A\cos\varphi - \lambda_2 A\sin\varphi\right)$$
$$-\mu\left(p^2+1\right)\left(q + \lambda_1 A\sin\varphi + \lambda_2 A\cos\varphi\right) - a\sin\omega t$$

Now the transformation (4.41) can be applied directly to the last equation:

$$q = u\sin\psi;\ q' = -\beta pu\sin\psi + p\sqrt{1-\beta^2}\,u\cos\psi \tag{4.46}$$

We obtain finally a system of the first order differential equations governing the new variables:

$$\begin{aligned}
A' &= \mu\left(u\sin\psi + A\left(1+\lambda_1\right)\sin\varphi + A\lambda_2\cos\varphi\right)\cos\varphi \\
&\quad + a\sin\omega t\cos\varphi \\
\varphi' &= 1-\mu\left(\frac{u}{A}\sin\psi + \left(1+\lambda_1\right)\sin\varphi + \lambda_2\cos\varphi\right)\sin\varphi \\
&\quad - \frac{a}{A}\sin\omega t\sin\varphi \\
u' &= -\beta pu - 2\mu\beta p\cos\psi\left[pu\left(-\beta\sin\psi + \sqrt{1-\beta^2}\cos\psi\right)\right. \\
&\quad \left. + \lambda_1 A\cos\varphi - \lambda_2 A\sin\varphi\right] - a\sin\omega t\cos\psi \\
&\quad - \mu\left(p^2+1\right)\cos\psi\left(u\sin\psi + A\lambda_1\sin\varphi + A\lambda_2\cos\varphi\right) \\
\psi' &= p\sqrt{1-\beta^2} + 2\mu\beta p\sin\psi\left[pu\left(-\beta\sin\psi + \sqrt{1-\beta^2}\cos\psi\right)\right. \\
&\quad \left. + \lambda_1 A\cos\varphi - \lambda_2 A\sin\varphi\right] + a\sin\omega t\sin\psi \\
&\quad + \mu\left(p^2+1\right)\sin\psi\left(u\sin\psi + A\lambda_1\sin\varphi + A\lambda_2\cos\varphi\right)
\end{aligned} \tag{4.47}$$

The obtained equations have the specific form similar to (4.19). The variable A is slow. The variables φ and ψ are the fast rotating phases with almost constant velocities. The variable u is the strongly damped one.

Let us consider the principal resonance in this system, which means the excitation frequency is close to the partial frequency of the large mass and the small mass is tuned to this frequency too:

$$\begin{aligned}
&\omega = 1+\delta;\ \delta << 1 \\
&p\sqrt{1-\beta^2} = 1+\varepsilon;\ \varepsilon << 1
\end{aligned} \tag{4.48}$$

In this case we can introduce the new variables meaning the corresponding phase differences:

$$\chi = \omega t;\ \alpha = \chi - \varphi;\ \theta = \chi - \psi \tag{4.49}$$

For these variables we can obtain the following equations:

$$
\begin{aligned}
A' &= \mu A\cos(\chi+\alpha)\big((1+\lambda_1)\sin(\chi+\alpha)+\lambda_2\cos(\chi+\alpha)\big)\\
&\quad +\mu u\sin(\chi+\theta)\cos(\chi+\alpha)+a\sin\chi\cos(\chi+\alpha)\\
\alpha' &= -\delta-\mu\sin(\chi+\alpha)\big((1+\lambda_1)\sin(\chi+\alpha)+\lambda_2\cos(\chi+\alpha)\big)\\
&\quad -\mu\frac{u}{A}\sin(\chi+\theta)\sin(\chi+\alpha)-\frac{a}{A}\sin\chi\sin(\chi+\alpha)\\
\theta' &= -\varepsilon+2\mu\beta pA\sin(\chi+\theta)\Big[\lambda_1\cos(\chi+\alpha)-\lambda_2\sin(\chi+\alpha)\\
&\quad +p\frac{u}{A}\Big(-\beta\sin(\chi+\theta)+\sqrt{1-\beta^2}\cos(\chi+\theta)\Big)\Big]\\
&\quad +\mu A\big(p^2+1\big)\sin(\chi+\theta)\big(\lambda_1\sin(\chi+\alpha)+\lambda_2\cos(\chi+\alpha)\big)\\
&\quad +\mu\big(p^2+1\big)u\sin(\chi+\theta)\sin(\chi+\theta)+a\sin\chi\sin(\chi+\theta)\\
u' &= -\beta pu\\
&\quad -2\mu\beta pA\cos(\chi+\theta)\Big[\lambda_1\cos(\chi+\alpha)-\lambda_2\sin(\chi+\alpha)\\
&\quad +p\frac{u}{A}\Big(-\beta\sin(\chi+\theta)+\sqrt{1-\beta^2}\cos(\chi+\theta)\Big)\Big]\\
&\quad -\mu A\big(p^2+1\big)\cos(\chi+\theta)\big(\lambda_1\sin(\chi+\alpha)+\lambda_2\cos(\chi+\alpha)\big)\\
&\quad -\mu\big(p^2+1\big)u\cos(\chi+\theta)\sin(\chi+\theta)-a\sin\chi\cos(\chi+\theta)\\
\chi' &= \omega
\end{aligned}
\tag{4.50}
$$

These are the equations in the required form (4.19). We have here the slowly oscillating variables A,α,θ, the fast rotating phase χ and the strongly damped (uncritical) fast variable u. The averaging procedure described in the section 4.1 can be applied to this system.

4.2.3. Equations of the First Order Approximation. Discussion of the Approach

Averaging the equations (4.50) one obtains a system in form (4.20). However we restrict the analysis to the first order approximation and omit the small terms. The equations can be split into two groups. The first one describes the motions of the centre of mass (variables A,α). The second one describes the "degenerated" slave motions (variables u,θ).

$$
\begin{aligned}
A' &= \frac{1}{2}\mu A\lambda_2 + \frac{1}{2}\mu u\sin(\theta-\alpha) - \frac{1}{2}a\sin\alpha \\
\alpha' &= -\delta - \frac{1}{2}\mu(1+\lambda_1) - \frac{1}{2}\mu\frac{u}{A}\cos(\theta-\alpha) - \frac{1}{2}\frac{a}{A}\cos\alpha \\
u' &= -\beta p u \\
\theta' &= -\varepsilon + \frac{1}{2}\mu A\left(p^2+1\right)\left(\frac{u}{A} + \lambda_1\cos(\theta-\alpha) + \lambda_2\sin(\theta-\alpha)\right) \\
&\quad + \mu\beta pA\left[\lambda_1\sin(\theta-\alpha) - \lambda_2\cos(\theta-\alpha) - p\frac{u}{A}\beta\right] \\
&\quad + \frac{1}{2}a\cos\theta
\end{aligned}
\tag{4.51}
$$

Index 1 for the first order approximation is omitted here.

It follows directly from the third equation that the stationary solution for the strongly damped slave variable u is equal to zero. It means that the variable q is also equal to zero in the first order approximation. So it does not make sense to calculate the variable θ describing the phase of these oscillations (the phase does not matter, if the amplitude is equal to zero). We can consider only the first two equations in determining the oscillations of the large mass:

$$
\begin{aligned}
A' &= \frac{1}{2}\mu A\lambda_2 - \frac{1}{2}a\sin\alpha \\
\alpha' &= -\delta - \frac{1}{2}\mu(1+\lambda_1) - \frac{1}{2}\frac{a}{A}\cos\alpha
\end{aligned}
\tag{4.52}
$$

The final equations (4.52) are very simple, but the method we used to obtain them seem to be rather complex as well, as the intermediate equations (4.50). We have used this way in order to demonstrate the formal approach. However we could easily avoid these complications if we have noticed that r is the strongly damped slave variable. It can be replaced through the particular enforced solution (4.44) and all the terms containing q could be omitted from the very beginning.

The formal approach is necessary if we cannot limit the analysis to the first order approximation and need the higher ones. Such an example can be found in Chapter 6.

The stationary solution to the system (4.52) can be obtained if we set the right-hand sides of these equations to zero and eliminate the phase difference α. The stationary amplitude depends on the difference between the natural and the excitation's frequencies and on the system's parameters as follows:

$$A = \frac{a}{\sqrt{\left(2\delta + \mu\left(1+\lambda_1\right)\right)^2 + \mu^2 {\lambda_2}^2}}$$
$$\lambda_1 = \frac{p^2 - 1}{\left(p^2 - 1\right)^2 + 4p^2\beta^2};\ \lambda_2 = -\frac{2p\beta}{\left(p^2 - 1\right)^2 + 4p^2\beta^2} \tag{4.53}$$

The considered principal resonance (4.48) means the following relationship between the parameters (with the accuracy of the first order):

$$p = \frac{1+\varepsilon}{\sqrt{1-\beta^2}} \tag{4.54}$$

4.2.4. Comparison with the Numeric Experiment. Discussion of the Results

Figure 4.2 shows the comparison between the approximate analytic prediction (4.53) and the direct numeric simulations of the original system (4.32). The calculations were performed for the following parameter values:

$$M = 1;\ C = 1;\ m = 0.01;\ c = 0.01;\ b = 0.005;\ a = 0.01;\ k = 1$$
$$\lambda = 1;\ \mu = 0.01;\ \beta = 0.25;\ p = 1;\ \lambda_1 = 0;\ \lambda_2 = -2;\ \varepsilon = -0.032$$

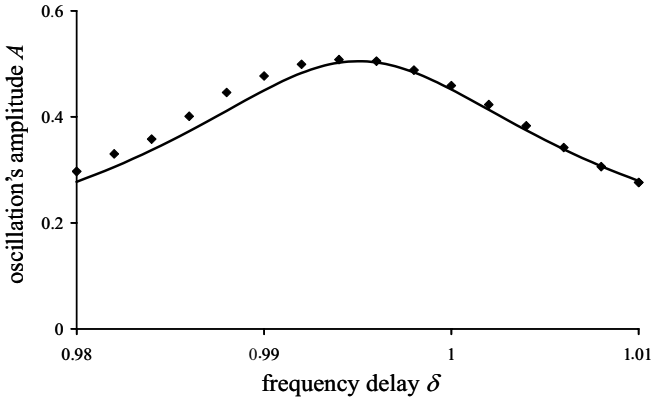

Fig. 4.2. Resonant amplitude as a function of the frequency delay; the solid line corresponds to the analytic prediction; the dots correspond to the results of the numeric simulations

The difference between the predictions is quite acceptable and corresponds to the asymptotic accuracy of the first order approximation.

It is also interesting to investigate how the resonant amplitude depends on the damping between the large and the small masses. The simplest result can be obtained if we suppose $p = 1$. It means the partial frequencies of the large and the small masses are equal. Then it is easy to see that $\lambda_1 = 0;\ \lambda_2 = -\dfrac{1}{2\beta}$ and the equation for the stationary amplitude can be simplified as follows:

$$A = \frac{a}{\sqrt{(2\delta + \mu)^2 + \dfrac{\mu^2}{4\beta^2}}} \tag{4.55}$$

It becomes evident from this equation that the resonant amplitude increases if we increase damping. Figure 4.3 illustrates this result for $\delta = 0$. All other parameters are the same as in the previous case.

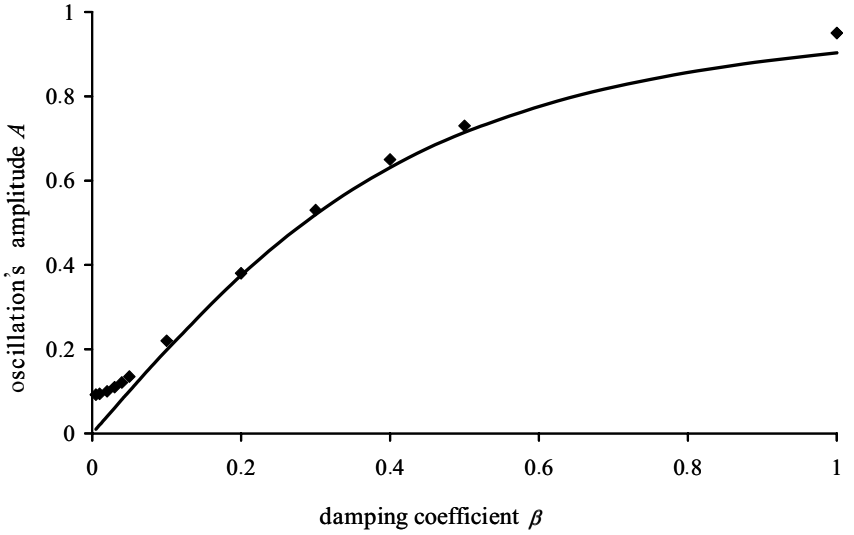

Fig. 4.3. Resonant amplitude as a function of the damping between the large and the small masses; the solid line corresponds to the analytic prediction; the dots correspond to the results of the numeric simulation

The correlation between the numeric and the asymptotic predictions remains acceptable. The discrepancy increases for small damping coefficients. This could be expected because the whole analysis is based on the assumption that the damping coefficient is not small. Only that's why we could set the variable u to zero.

The result itself, however, seems to be unexpected. It is common that increasing damping reduces the resonant amplitude. Here we see the opposite effect. Its physical explanation is simple.

We have considered the amplitude for the exact resonance of the large mass (not for the whole frequency range as it is usual for dynamic dampers). The large damping connects the large and the small masses. We strengthen this connection increasing the damping. The infinite damping means nothing different but the rigid connection of the masses. In this case we would obtain a linear oscillator without any damping. In other words the strong damping prevents intensive relative movements between the masses. This is the reason for the apparent paradox.

This is also the physical reason for the "slavery" of the small mass. It is very important to notice that the strongly damped "slave" does not have its own dynamics. Due to the strong damping it can only follow the motions of the "master". We will see the same behavior in the next paragraph, where the source of the intensive energy dissipation is not the linear damping but absolutely inelastic collisions between the masses.

4.3. Averaging in Systems with Inelastic Collisions: Basic Ideas and General Approach

4.3.1. Basic Types of Motion in Systems with Inelastic Collisions: Elementary Examples

An inelastic collision of one-dimensional rigid bodies means that their velocities after the collision are equal. It does not mean however that the two bodies necessarily move together after the collision. In order to see it consider the following example shown in Fig. 4.4.

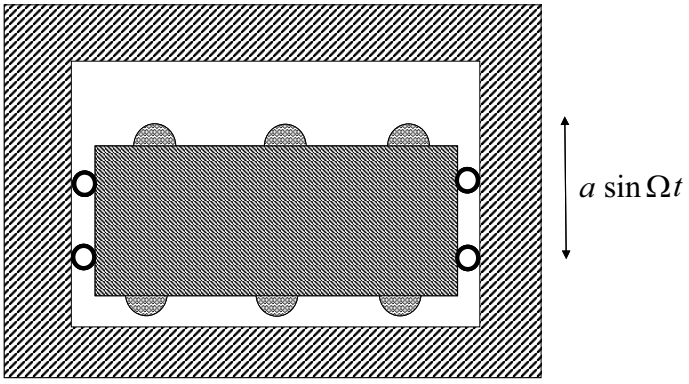

Fig. 4.4. A rigid oscillating frame with a free mass inside

This system consists of a rigid harmonically oscillating frame and a free mass inside of it. This system is the simplest model for a stone crasher. The frame represents the oscillating housing of the machine and the free mass corresponds to the crashing body. Fig. 4.5 and 4.6 show the two qualitative different types of motions of the free mass after a collision. It is characteristic for the first type of motion that the colliding bodies (the mass and the frame) move together for a certain finite time interval.

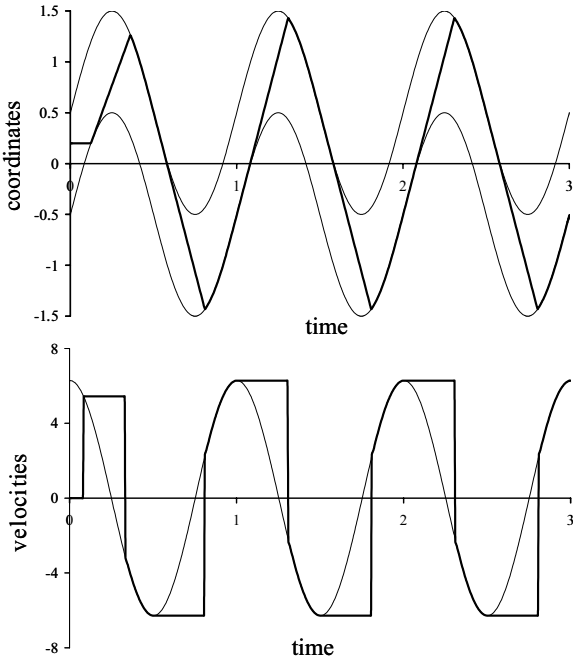

Fig. 4.5. Motion of the mass in the oscillating frame with long time intervals of contact; thin lines correspond to motion of the frame; thick lines correspond to motion of the free mass

The second type of motion is characterized by the fact that the time interval of the contact between the colliding bodies (the mass and the frame) is infinitesimally short. The physical difference between these two types of collisions is very simple. In the first case the contact force after the collision remains positive. In the second case the contact force after the collision would be negative if we suppose the contact to continue. It means the immediate contact lost after the equalizing of the velocities.

It is not difficult to calculate the contact force in this case. For the contact with the bottom of the frame one obtains:

$$F_{contact}^{bottom} = m\ddot{x}; x = a\sin\Omega t \Rightarrow F_{contact}^{bottom} = -ma\Omega^2 \sin\Omega t \qquad (4.56)$$

The equation for the contact with the top of the frame is similar:

$$F_{contact}^{top} = -m\ddot{x}; x = a\sin\Omega t \Rightarrow F_{contact}^{top} = ma\Omega^2 \sin\Omega t \qquad (4.57)$$

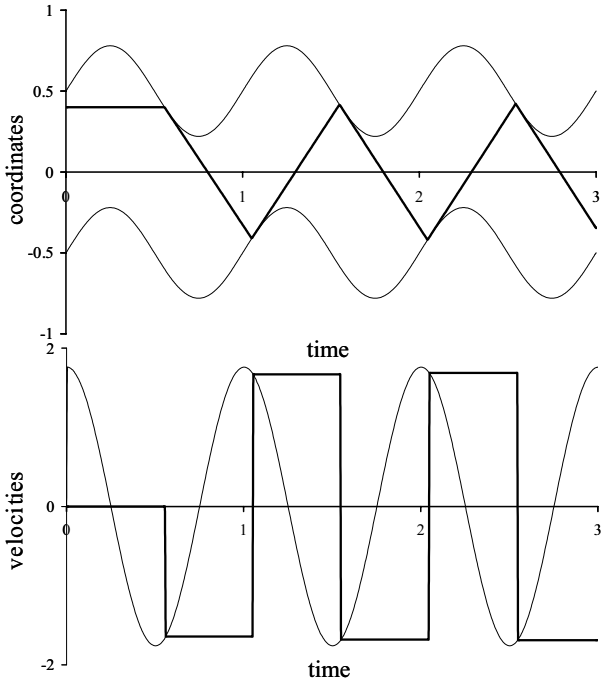

Fig. 4.6. Motion of the mass in the oscillating frame with infinitesimally short time interval of contact; thin lines correspond to motion of the frame; thick lines correspond to motion of the free mass

It means the contact with the bottom of the frame after a collision is possible as long as the acceleration of the frame is positive. The contact with the top of the frame is possible as long as the acceleration of the frame is negative. If these conditions are not fulfilled to the time of collision the mass looses the contact immediately.

Figures 4.5 and 4.6 show simple periodic motions with long contact and infinitesimally short contact respectively. The mass collides with each side of the frame once per each oscillations period. These are the simplest oscillation regimes in this system but different much more complex regimes are also possible. Figure 4.7 shows a periodic regime where the mass collides with each side of the frame once per three oscillation periods.

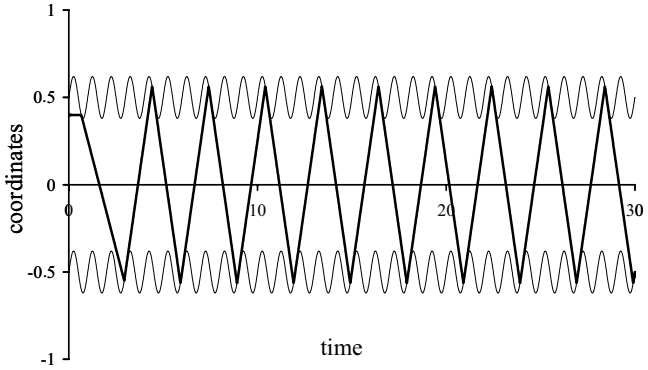

Fig. 4.7. Periodic regime with one collision at each side of the frame per three oscillation periods

Figure 4.8 shows a periodic regime with different types of motions after the collisions with the top and with the bottom of the frame. Figure 4.9 shows a complex periodic regime.

These examples demonstrate that even though the considered system seems to be very simple, its dynamic behavior can be quite different depending on the parameter values. But all these regimes are based on the described two types of collisions:

- An inelastic collision with a long contact;
- An inelastic collision with the infinitesimally short contact.

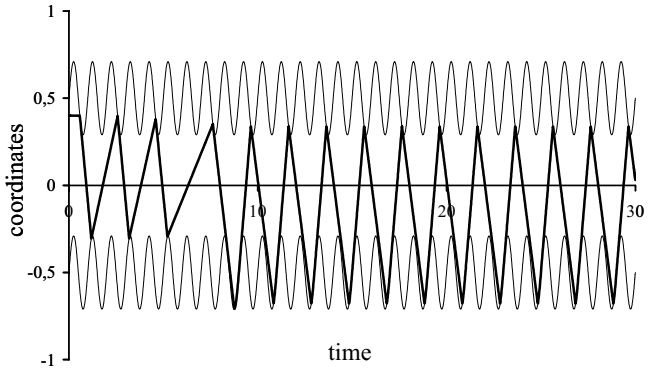

Fig. 4.8. Oscillations of the mass in the frame; the long contact follows to each collision with the bottom of the frame; the contact after each collision with the top of the frame is infinitesimally short

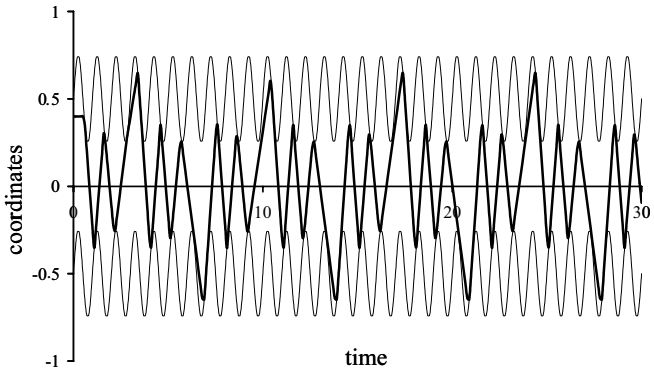

Fig. 4.9. Complex oscillations of the mass in the frame with 10 collisions in 7 frame's oscillation periods (one collision at each side is with a long contact and four collisions are with short contacts)

These two basic regimes can be also found in the second example shown in Fig. 4.10. It is a mass in the gravity field over an oscillating base. This system is the simplest model for a vibrating transporter or a vibrating screen. The mass corresponds usually to the layer of a bulk material.

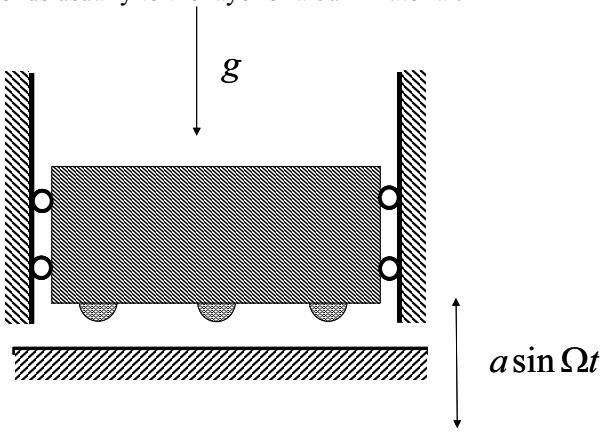

Fig. 4.10. A free mass over the harmonically oscillating base

The contact force can be calculated as follows:

$$F_{contact} = m\ddot{x} + mg; x = a\sin\Omega t \Rightarrow F_{contact} = m\left(g - a\Omega^2 \sin\Omega t\right) \quad (4.58)$$

Fig. 4.11 shows an oscillations' regime with long contacts after each collision.

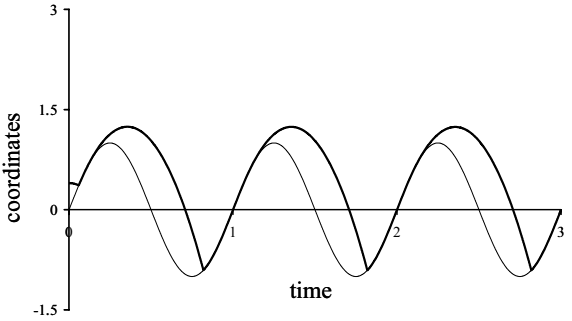

Fig. 4.11. Motion of the mass over the oscillating base with long time intervals of contact; the thin line corresponds to the motion of the base; the thick line corresponds to the motion of the free mass

A contact is long if the contact force is positive directly after the collision. It means that the time point of collision must fulfill the following condition:

$$\begin{aligned}&(2n-1)\pi-\arcsin\frac{g}{a\Omega^2}\le\Omega t_{collision}^{long}<\arcsin\frac{g}{a\Omega^2}+2\pi n\\&n=\ldots-2,-1,0,1,2\ldots\end{aligned}\tag{4.59}$$

If these conditions are not fulfilled, the only the infinitesimally short contact is possible (see Fig. 4.12):

$$\begin{aligned}&\arcsin\frac{g}{a\Omega^2}+2\pi n\le\Omega t_{collision}^{short}<(2n+1)\pi-\arcsin\frac{g}{a\Omega^2}\\&n=\ldots-2,-1,0,1,2\ldots\end{aligned}\tag{4.60}$$

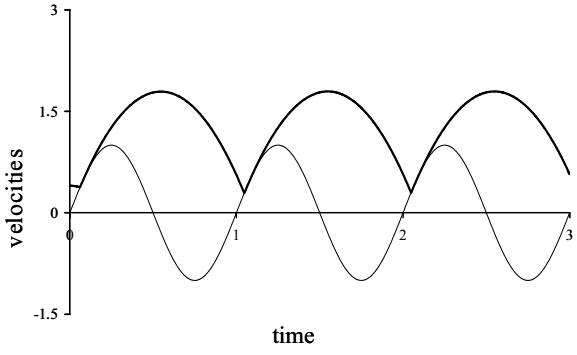

Fig. 4.12. Motion of the mass over the oscillating base with infinitesimally short time intervals of contact

The physical meaning of these conditions is quite simple. The mass attains the velocity of the base direct after the collision. This velocity remains constant if there is no external force. In presence of the external force the mass would decelerate (fall down) if the contact would not exist. The base moves according to its oscillation's law. This means that its velocity doesn't remain constant. If the deceleration of the mass is larger than the deceleration of the base the mass will be pressed towards the base and the long contact will be established. If the base decelerates faster the mass will loose the contact immediately. This means the "short" contact.

Complex regimes with changing types of contact are also possible in the last system. Fig. 4.13 shows such an example.

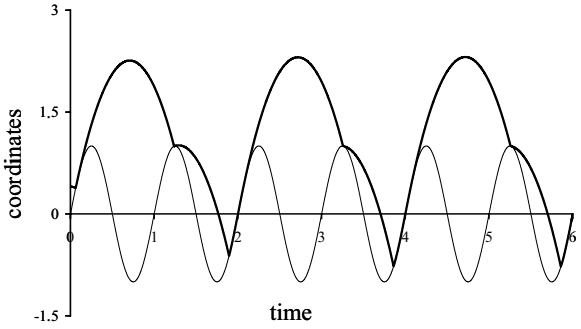

Fig. 4.13. Complex oscillations of the mass over the base with changing types of collisions

4.3.2. On the Practical Importance of Regimes with Long Contact

The variety of dynamic regimes in the considered examples shows that motions in systems with inelastic collisions can be very complex. It is seldom possible to investigate an initial value problem in these systems analytically. In many practical cases however it is not necessary to investigate the general solutions. The desired regime is often clear from the technological point of view. The real task in such a situation is to choose the system's parameters guaranteeing existence and stability of the required dynamic regime and (if it is necessary) to create the appropriate control system.

The technological processes (screening or crashing) take place usually during the collision. It is sensible from that point of view to organize the collisions with the highest possible frequency and intensity. Another important criterion is the robustness of the dynamic regime with respect to inaccuracies and perturbation of all possible kinds (like inaccuracy of the system's parameters, undetermined initial conditions and unexpected external influences).

Let us consider the mass in the frame. The highest frequency of the intensive collisions can be achieved if the mass collides once per oscillation's period with each side of the frame. It is sensible to require the highest possible velocity of the

mass in order to intensify the collisions. It becomes clear looking at the Fig. 4.5 that the highest velocity of the free mass can be reached only at the end of the long contact phase. That's why the simplest dynamic regime with two long contacts per oscillation's period is of the most interest for applications. On the other hand the velocity of the frame should also be as large as possible (and have the opposite direction with respect to the velocity of the free mass). It means the collision should take place as close as possible to the end of the extreme point of the frame's velocity, but this point separates the regimes with long and short contacts. Exactly this boundary would be of the most technological interest.

The regime with long contacts is also the best one because it is very robust and stable. The mass looses the contact with the frame when the contact force becomes equal to zero. Then it moves away from the frame with initial position and velocity determined by the motion of the frame only. These initial conditions don't depend on the whole prehistory of the system. In other words the system forgets its history as soon as the long contact occurs.

Even large slow changes in the excitation's amplitude often do not change the structure of this regime (the number and type of collisions in each oscillation's period). This statement is illustrated in Fig. 4.14.

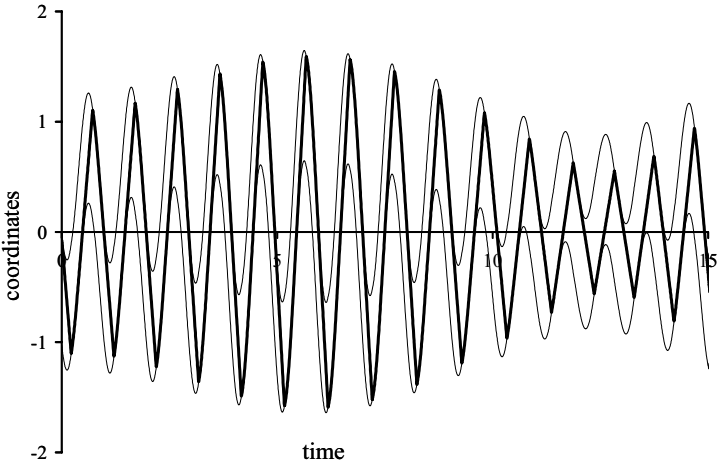

Fig. 4.14. Slow variations of the amplitude of the frame don't change the structure of the basic regime with long contacts

The situation with vibrating screens is similar. Regimes with long contacts are very robust. The intensity of collisions however increases with the increasing velocity. From this point of view also the regimes with one long collision per several oscillations of the base can be technologically interesting.

These arguments explain the exceptional role of the simple regimes with long contacts for applications. That's why the analysis in this and the next sections is concentrated on the simple regimes with long contact.

4.3.3. Regimes with Long Contacts as an Example of the Variable Order Discontinuous Systems

Let us consider the mass in the frame and introduce the coordinate system connected with the frame. It means that we describe the motion of the mass relative to the oscillating frame. Then the motion of the mass is governed by the following equations (r is the coordinate of the mass; $2l$ is the length of the frame).

During the contact phases the coordinate of the mass doesn't change. It means

$$\ddot{r} = 0 \text{ as long as } r = \pm l \text{ and } F_{contact} = ma\Omega^2 \operatorname{sgn}(r)\sin\varphi > 0 \tag{4.61}$$

Compare these relationships with the contact conditions (4.56) and (4.57).

Here and further we use the following notation

$$\Omega t = \varphi \tag{4.62}$$

During the free flight the absolute velocity remains constant:

$$\ddot{x} = 0 \Rightarrow \ddot{r} = a\Omega^2 \sin\varphi \text{ as long as } |r| < l \tag{4.63}$$

The two equation sets must be linked together through the appropriate initial conditions for the time (or the phase) corresponding to the beginning of the free flight $t_0 \leftrightarrow \varphi_0$ and for the time of collision $t_1 \leftrightarrow \varphi_1$. These values can be determined as follows.

$$\begin{aligned} &\sin\varphi_0 = 0 \qquad \Rightarrow \qquad \varphi_0 = \left(2n + \frac{1+\operatorname{sgn} r}{2}\right)\pi \\ &r(t_0) = \pm l; \dot{r}(t_0) = 0 \end{aligned} \tag{4.64}$$

$$\varphi_1 = \varphi_0 + \alpha: \quad r = \pm l; \dot{r} = 0 \tag{4.65}$$

Parameter α is the first positive solution of the transcendent equation

$$\alpha - \sin\alpha = \frac{2l}{a} \tag{4.66}$$

The motion of the mass in these variables is shown in Fig. 4.15 (it corresponds to the regime in Fig. 4.5)

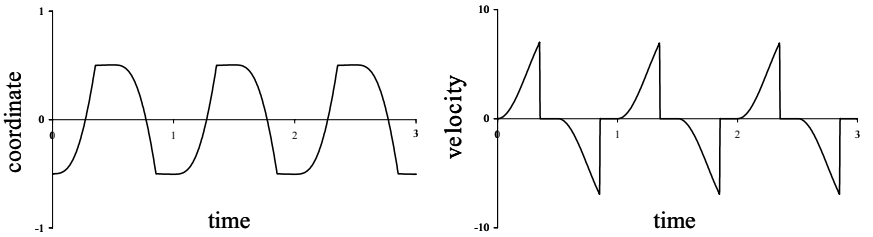

Fig. 4.15. Motion of the free mass in the oscillating frame; the relative coordinate r and the relative velocity $\dot{r}$

The considered regime is a symmetric one with respect to motions up and down. So it is sensible to introduce the new coordinate and velocity as follows

$$\begin{aligned} u &= r \operatorname{sgn} \sin \varphi \\ v &= \dot{r} \operatorname{sgn} \sin \varphi \end{aligned} \tag{4.67}$$

The motion in these phase coordinates is shown in Fig. 4.16.

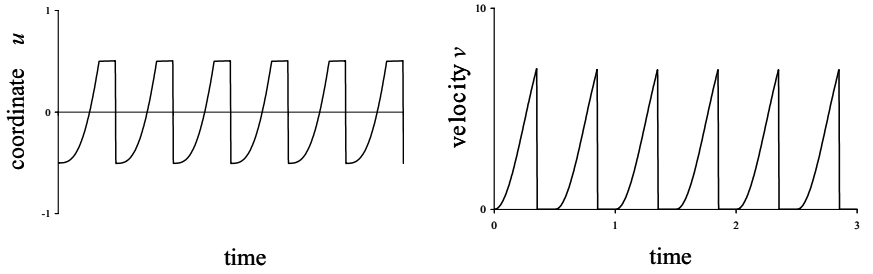

Fig. 4.16. The same solutions for the coordinate u and the velocity v

The solution for these variables can be formalized quite easily. Let us introduce the following step function indicating if the mass is in contact with the frame or flying up or down (see Fig. 4.17)

$$M_1(\varphi,\alpha) = \begin{cases} 1, & 0 < \varphi < \alpha \\ 0, & \alpha < \varphi < 2\pi \end{cases}; \quad M_1(\varphi + 2\pi, \alpha) = M_1(\varphi,\alpha) \tag{4.68}$$

Now we can apply the transformation combining (4.67) with the indicator-function:

$$\begin{aligned} M_2(\varphi,\alpha) &= M_1(2\varphi,\alpha) \operatorname{sgn} \sin \varphi \\ r &= -l \operatorname{sgn} \sin \varphi + U M_2(\varphi,\alpha) \\ \dot{r} &= W M_2(\varphi,\alpha) \end{aligned} \tag{4.69}$$

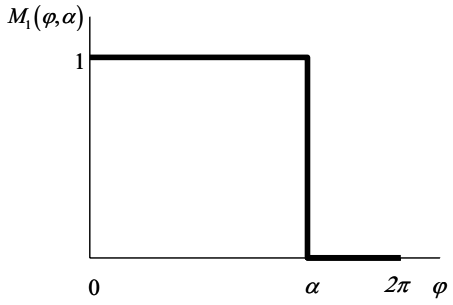

Fig. 4.17. Indicator-function

This transformation fulfils the equations (4.61) during the long contact phases automatically. Applying this transformation we can rewrite the equations (4.61)–(4.66) as follows:

$$\begin{aligned}
&\Omega t \neq n\pi: \quad \begin{cases} M_2(\Omega t,\alpha)\dot{U} = M_2(\Omega t,\alpha)W \\ M_2(\Omega t,\alpha)\dot{W} = M_2(\Omega t,\alpha)a\Omega^2 \sin\Omega t \end{cases} \\
&\Omega t = n\pi: \quad U = 0;\ \dot{U} = 0 \\
&\alpha - \sin\alpha = \frac{2l}{a}
\end{aligned} \tag{4.70}$$

Relationships (4.70) do not determine a system of ordinary differential equation. It is something different. It is an infinite sequence of systems of ordinary differential equations. Each of them is valid during the time interval $\pi n < \Omega t < \pi n + \alpha$ because during these time intervals the function $M_2(\Omega t,\alpha)$ is equal to 1 or –1. During the next time interval $\pi n + \alpha < \Omega t < \pi(n+1)$ there are no equations at all because the function $M_2(\Omega t,\alpha)$ is equal to zero (the equations are degenerated to an identity). At the beginning of the next time interval of the non-degenerate motion the new initial conditions must be determined.

The solution to the system (4.70) is:

$$\pi n < \Omega t < \pi n + \alpha: \begin{cases} W = a\Omega\left(1 - \cos\Omega t \cdot \operatorname{sgn}\sin\Omega t\right) \\ U = a\Omega\left(t - \pi\left[\dfrac{\Omega t}{\pi}\right]\right) - a\left|\sin\Omega t\right| \end{cases} \tag{4.71}$$

$\pi n + \alpha < \Omega t < \pi(n+1)$: U, W are not defined

The sign $\left[z\right]$ means here and further the largest integer number smaller than z. Returning back to the original variables we can write:

$$r = l\,\mathrm{sgn}\sin\Omega t\left\{1 - M_1\left(2\Omega t,\alpha\right)\right\} - M_1\left(2\Omega t,\alpha\right)a\sin\Omega t + \left\{-l + a\Omega\left(t - \pi\left[\frac{\Omega t}{\pi}\right]\right)\right\}M_2\left(\Omega t,\alpha\right) \tag{4.72}$$

$$\dot{r} = a\Omega\left(\mathrm{sgn}\sin\Omega t - \cos\Omega t\right)M_1\left(2\Omega t,\alpha\right)$$

The first term in the equation for r corresponds to the position of the mass during contact, i.e. when $M_1\left(2\varphi,\alpha\right) = 0$.

Equations (4.70) are the simplest example of the variable order discontinuous system. Within the time intervals where the function M_2 differs from zero it is a system of the second order. Within the time intervals where the function M_2 is equal to zero we don't have any differential equation (order zero). The length of each time interval is finite and at the beginning of the next time interval the system obtains new initial conditions.

4.3.4 Variable Order Discontinuous Systems in the Standard Form

An averaging procedure can be developed for the variable order discontinuous systems. Let us firstly define the corresponding standard form.
Consider a system described over certain times by differential equations, and at other time intervals by differential and finite relations of the following form:

$$\begin{aligned} &\dot{x} = \varepsilon X\left(x, yM\left(t\right), t, \varepsilon\right);\ x\left(0\right) = x_0 \\ &\dot{y}M\left(t\right) = \varepsilon Y\left(x, yM\left(t\right), t, \varepsilon\right)M\left(t\right); \\ &y\left(2\pi n\right) = G\left(x\left(2\pi n\right), \varepsilon\right),\ \ n = 0,1,2,\ldots \end{aligned} \tag{4.73}$$

Here $M\left(t\right) = M_1\left(t,\pi\right)$ is a 2π-periodic piecewise-constant function. Vector x contains the normal slow variables and describes the subsystem of the constant order. The vector-function y is a solution of an infinite sequence of systems of differential equations and describes the discontinuous subsystem of the variable order. The equations for y are valid within the time intervals where the function M is not equal to zero. Where the function M is equal to zero there are only equations for slow continuous variables x. New initial conditions for y are given at the beginning of each interval where y is defined. These initial condi-

tions may depend on the values reached by the variables x to the end of the previous interval.

The main idea for the averaging of the system (4.73) is that the variables y are the "slaves" of the master variables x (at least in the first order approximation). Let us assume that we know $x(2\pi n)$ for some certain n (for example the initial condition for $n=0$). Then we also know the corresponding initial condition for $y(2\pi n)=G(x(2\pi n))=G_n$. The equations for y must be integrated only as long as the function M is not equal to zero. But this interval is asymptotically short and the variable y is slow. So it cannot change significantly during this time interval and we can replace it through its initial value or with the same accuracy we can write:

$$y = G_n + O(\varepsilon) = G(x) + O(\varepsilon) \tag{4.74}$$

Substituting (4.74) into the equation for x we get a system of ordinary differential equations in the standard form:

$$\dot{x} = \varepsilon X\left(x, G(x,\varepsilon)M(t), t, \varepsilon\right) + O(\varepsilon^2);\ x(0) = x_0 \tag{4.75}$$

This system can be averaged directly. Finally the variable y can be calculated according to (4.74).

In other words, consider the system (4.73) alongside the averaged one

$$\begin{aligned} \dot{\xi} &= \varepsilon \left\langle X\left(\xi, G(\xi,\varepsilon)M(t), t, \varepsilon\right)\right\rangle \\ \eta &= G(\xi,\varepsilon);\quad \xi(0) = x_0 \end{aligned} \tag{4.76}$$

The solutions to these two systems remain asymptotically close to each other for the asymptotically long time:

$$\begin{aligned} &\|x-\xi\| \le O(\varepsilon) \\ &\|yM(t)-\eta M(t)\| \le O(\varepsilon)M(t) \end{aligned} \quad \text{for } t \le O(1/\varepsilon) \tag{4.77}$$

The correct mathematical formulation of the corresponding theorem and its proof can be found in Appendix VI.

The switching points of the function M were constant in the considered case. This requirement is not important. The principal estimation (4.77) doesn't change if we replace the function $M(t)$ through $M_1(t,\alpha)$ and even if we suppose that α is not a constant but it depends on the slow variables (see Appendix VI for the corresponding theorem and the proof).

4.4. Basic Regime with Long Contacts for the Mass in a Resonantly Excited Frame

4.4.1. Equations of Motion

The elementary scheme of the resonant impact crasher is shown in Fig.4.18.

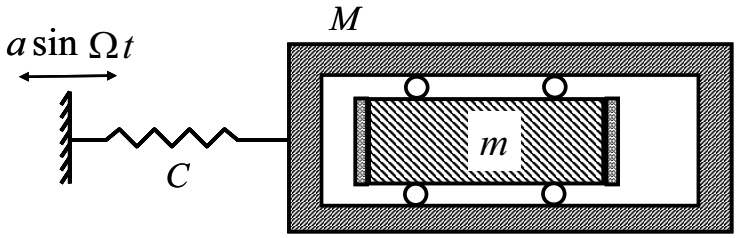

Fig. 4.18. Impact crasher with resonant excitation

It consists of a frame with a large mass M excited kinematically. Inside of the frame there is a free moving striker. Its mass is m. The free length of the frame's interior is $2l$. The striker can collide with both sides of the frame. The collisions are absolutely inelastic.

Let us start with the equations for the time intervals of separate motion of the masses. x_1 is the coordinate of the frame, x_2 is the coordinate of the striker.

$$\begin{aligned} &M\ddot{x}_1 + C\left(x_1 - a\sin\Omega t\right) = 0 \\ &m\ddot{x}_2 = 0 \end{aligned}, \quad \left|x_2 - x_1\right| < l \tag{4.78}$$

During the contact phase the frame and the striker move together:

$$\begin{aligned} &\left(M+m\right)\ddot{x}_1 + C\left(x_1 - a\sin\Omega t\right) = 0 \\ &\ddot{x}_2 = \ddot{x}_1 \end{aligned}, \quad \left|x_2 - x_1\right| = l \tag{4.79}$$

The contact is possible as long as the contact force remains positive.

$$F_{contact} = -\frac{m}{M+m}C\left(x_1 - a\sin\Omega t\right)\mathrm{sgn}\left(x_2 - x_1\right) \tag{4.80}$$

It is sensible to introduce the coordinates corresponding to the position of the center of mass of the system and to the distance between the frame and the striker:

$$s = \frac{Mx_1 + mx_2}{M+m};\ r = x_2 - x_1 \tag{4.81}$$

The new variables are governed by the following equations (compare (4.30)-(4.32))

$$\begin{aligned} s'' &= -(1-\mu)s + \mu(1-\mu)r + a(1-\mu)\sin\omega\tau \\ r'' &= \begin{cases} s - \mu r - a\sin\omega\tau, & |r| < l \\ 0, & |r| = l \end{cases} \end{aligned} \tag{4.82}$$

These equations must be supplemented by the corresponding conditions for the separation of the contact:

$$|r| = l : s = \mu r + a\sin\omega\tau;\ \dot{r} = 0 \tag{4.83}$$

The following notation used in these equations:

$$\tau = kt;\ k = \sqrt{\frac{C}{M}};\ \mu = \frac{m}{M+m};\ \omega = \frac{\Omega}{k};\ (\ )' = \frac{d(\ )}{d\tau} \tag{4.84}$$

Assuming that μ and a/l are small parameters of the same order we can linearize equations (4.82) with respect to these parameters:

$$\begin{aligned} s'' + s &= \mu s + \mu r + a\sin\omega\tau \\ r'' &= \begin{cases} s - \mu r - a\sin\omega\tau, & |r| < l \\ 0, & |r| = l \end{cases} \end{aligned} \tag{4.85}$$

This is the system we are going to analyze.

4.4.2. Perturbation Analysis. Transformation to the Form Suitable for Averaging

Consider the unperturbed system:

$$\begin{aligned} s_0'' + s_0 &= 0 \\ r_0'' &= \begin{cases} s_0, & |r_0| < l \\ 0, & |r_0| = l \end{cases} \end{aligned} \tag{4.86}$$

The general solution to the first equation is obvious:

$$s_0 = A_0 \sin\varphi_0;\ A_0 = const;\ \varphi_0 = \tau + const \tag{4.87}$$

Substituting this solution into the second equation we obtain the system which was analyzed in section 4.3.3.

$$r_0'' = \begin{cases} A_0 \sin\varphi_0, & |r_0| < l \\ 0, & |r_0| = l \end{cases} \tag{4.88}$$

The corresponding solution is known, so we can use it as the variable transformation in order to obtain a system suitable for averaging:

$$\begin{aligned} &s = A\sin\psi;\ s' = A\cos\psi;\ \chi = \omega\tau \\ &r' = \left\{B\,\mathrm{sgn}\sin\varphi - A\cos\psi\right\} M_1\left(2\varphi,\alpha\right) \\ &r = l\,\mathrm{sgn}\sin\varphi\left\{1 - M_1\left(2\varphi,\alpha\right)\right\} - M_1\left(2\varphi,\alpha\right) A\sin\psi \\ &\quad + \left\{-l + B\left(\varphi - \pi\left[\frac{\varphi}{\pi}\right]\right)\right\} M_2\left(\varphi,\alpha\right) \end{aligned} \tag{4.89}$$

The transformation (4.89) is based on the solution (4.72). The newly introduced variable B has the sense of the velocity of the striker when it looses the contact with the frame. Parameter α determines the time of the next collision:

$$t_{collision}: \begin{cases} \varphi = \alpha \\ B\alpha - A\sin\psi = 2l \end{cases} \tag{4.90}$$

This transformation must be applied both to the equations (4.85) and the conditions for the separation of the contact (4.83). The new set of variables is governed by the following equations:

$$\begin{aligned} &A' = \mu A\sin\psi\cos\psi + \mu r\cos\psi + a\sin\chi\cos\psi \\ &\psi' = 1 - \mu\sin^2\psi - \mu\frac{r}{A}\sin\psi - \frac{a}{A}\sin\chi\sin\psi \\ &\chi' = \omega \\ &B' M_1\left(2\varphi,\alpha\right) = \mu A\sin\psi\, M_1\left(2\varphi,\alpha\right)\mathrm{sgn}\sin\varphi \\ &\varphi' M_1\left(2\varphi,\alpha\right) = M_1\left(2\varphi,\alpha\right) \\ &\qquad - \mu\left(\varphi - \left[\frac{\varphi}{\pi}\right]\right)\frac{A}{B}\sin\psi\,\mathrm{sgn}\sin\varphi M_1\left(2\varphi,\alpha\right) \end{aligned} \tag{4.91}$$

$$\varphi = n\pi \Rightarrow B = A + O\left(\mu^2\right);\ \psi = n\pi + O\left(\mu\right) \tag{4.92}$$

Let us investigate the main resonance. This means that we, as usual, assume that the excitation's frequency is close to the natural frequency of the frame and introduce the corresponding phase difference:

$$\omega - 1 = \Delta << 1;\ \theta = \chi - \psi \tag{4.93}$$

Finally we obtain:

$$\begin{aligned}
&A' = \mu A \sin\psi \cos\psi + \mu r \cos\psi + a \sin\left(\psi + \theta\right)\cos\psi \\
&\theta' = \Delta + \mu \sin^2\psi + \mu \frac{r}{A}\sin\psi + \frac{a}{A}\sin\left(\psi + \theta\right)\sin\psi \\
&B' M_1\left(2\varphi, \alpha\right) = \mu A \sin\psi M_1\left(2\varphi, \alpha\right)\operatorname{sgn}\sin\varphi \\
&\varphi' M_1\left(2\varphi, \alpha\right) = M_1\left(2\varphi, \alpha\right) \\
&\qquad - \mu\left(\varphi - \left[\frac{\varphi}{\pi}\right]\right)\frac{A}{B}\sin\psi \operatorname{sgn}\sin\varphi M_1\left(2\varphi, \alpha\right) \\
&\psi' = 1 - \mu \sin^2\psi - \mu \frac{r}{A}\sin\psi - \frac{a}{A}\sin\left(\psi + \theta\right)\sin\psi
\end{aligned} \tag{4.94}$$

The system (4.94), (4.92) contains two slow master variables A and θ, one fast rotating phase ψ and two slave variables B and φ. The fact that the slave variable φ is fast is not really important, because its velocity is close to one. It means the variable $\left(\varphi - \psi\right) M_1\left(2\varphi, \alpha\right)$ is slow in its definition area. The averaging procedure described in the section 4.3.4 can be applied here.

4.4.3. Equations of the First Order Approximation. Discussion of the Results

Averaging the system (4.94), (4.92) over the fast phase we obtain the equations of the first order approximation.

$$A_1' = \frac{1}{2} a \sin\theta_1 - \frac{4\mu l}{\pi} \frac{\sin^4\left(\frac{\alpha_1}{2}\right)}{\alpha_1 - \sin\alpha_1}$$

$$\theta_1' = \Delta + \frac{\mu}{2} + \frac{a}{4l}\left(\alpha_1 - \sin\alpha_1\right)\cos\theta_1 + \frac{\mu}{\pi}\left(\sin\alpha_1 - \frac{1}{2}\sin 2\alpha_1 - \frac{\alpha_1}{2}\right) \tag{4.95}$$

$$\alpha_1 - \sin\alpha_1 = \frac{2l}{A_1};\ B_1 = A_1;\ \varphi_1 = \psi_1$$

Index 1 indicates here as usual the first order approximation.

The first three equations govern the motion of the mass point of the system (the frame). Setting the right hand sides in the first two equations to zero and considering the result together with the third equation we obtain a system of transcendental equations describing the stationary resonant regime. Figure 4.19 shows the resonant curve. The simulations were performed for $\mu = 0.1; a/l = 0.2; l = 0.5$

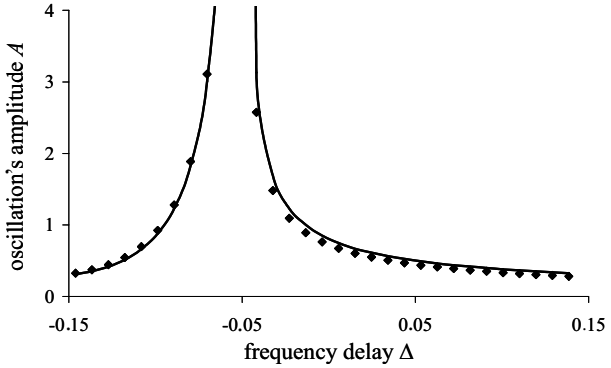

Fig. 4.19. Resonant amplitude as a function of the frequency delay; the solid line corresponds to the analytic prediction; the dots show the results of numeric simulations

The obtained solution corresponds to the simplest regime with one collision at each side of the frame per one oscillation period. Mathematically it means that the following condition must be fulfilled:

$$0 < \alpha_1 < \pi \tag{4.96}$$

The case $\alpha_1 = \pi$ corresponds to the transition from the regime with long contacts to the regime with short contacts. This condition limits the acceptable frequency delay:

$$-\frac{\pi a}{4l} < \Delta < \frac{\pi a}{4l} \tag{4.97}$$

The case $\alpha_1 = 0$ corresponds to exact resonance: $\Delta = -\mu/2$. It is interesting to point out that the dissipative influence of the inelastic impacts decreases in the vicinity of the resonance and the stationary amplitude at the exact resonance is infinite. This strange result becomes clear if we notice that at the exact resonance the striker flies from one side of the frame to another one in a very short time. It reaches the other side with a velocity the same as its own. So their relative velocity at the moment of the collision is very small and so is the lost energy. Consequently inelastic collisions can't limit the amplitude at the resonance. It is interesting to calculate the impact intensity which is the most important characteristic from the technological point of view. The relative velocity at the moment of collision is the sensible measure for the intensity. According to the obtained solution it can be calculated in the non-dimensional form as follows (see also Fig. 4.20):

$$I = \frac{A_1}{2l}\left(1 - \cos\alpha_1\right) = \frac{1 - \cos\alpha_1}{\alpha_1 - \sin\alpha_1} \tag{4.98}$$

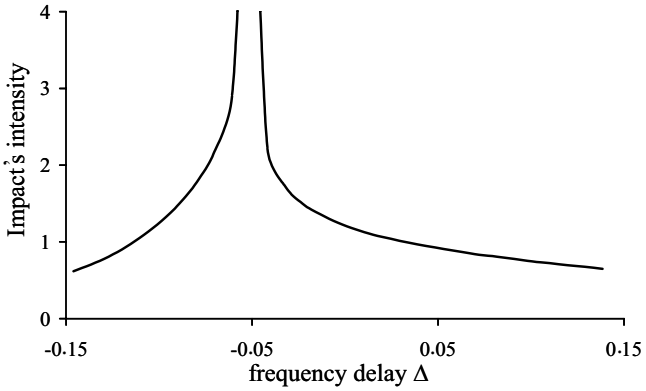

Fig. 4.20. Impact intensity over frequency delay

Another important point which has to be discussed is the sense of the first order approximation (4.95) compared to the unperturbed problem considered in the section 4.3.3. The last two equations obtained for the "slave variables" in (4.95) show that the motion of the striker doesn't differ from the particular solution for the unperturbed system. This means that actually we could replace the slave degrees of freedom through the corresponding enforced solution from the very beginning. Then the whole analysis would deal with the resonant motions of the "continuous"

subsystem taking the relative motions of the strike into account in the small perturbation terms on the right hand sides of the equations.

From that point of view we have here the same situation as in the strongly damped linear system considered at the beginning of this chapter. The degrees of freedom connected with the slave variables can be neglected and replaced through the corresponding particular solutions (depending on the motion of the master subsystem). Then the master subsystem can be averaged and the result can be substituted in the equations for the "slave" variables.

Using this approach one neglects the initial transient motions (until the first long contact) in the "slave" subsystem. The slow transient motions in the master system however can be investigated using the averaged master equations.

This approach is illustrated in the next section.

4.5. The Basic Regime with Long Contacts for the Mass over the Resonantly Excited Base

4.5.1. Equations of Motion

The elementary scheme of the resonant impact crasher is shown in Fig.4.21.

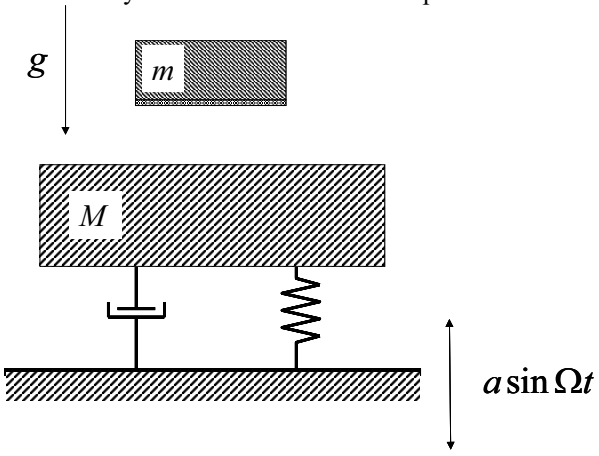

Fig. 4.21. Vibrating screen with resonant excitation

It consists of a base with a large mass M excited kinematically. A free mass m moves above the base in the gravity field g. The free mass can collide with the oscillating base. The collisions are absolutely inelastic.

Let us start with the equations for the time intervals of separate motion of the masses. x_1 is the coordinate of the base, x_2 is the coordinate of the free mass.

$$\begin{aligned} &M\ddot{x}_1 + b\left(\dot{x}_1 - a\Omega\cos\Omega t\right) + C\left(x_1 - a\sin\Omega t\right) = -Mg \\ &m\ddot{x}_2 = -mg \end{aligned} \quad , x_2 > x_1 \qquad (4.99)$$

During the contact phase the base and the layer move together:

$$\begin{aligned} &\left(M+m\right)\ddot{x}_1 + b\left(\dot{x}_1 - a\Omega\cos\Omega t\right) + C\left(x_1 - a\sin\Omega t\right) = \\ &\qquad -\left(M+m\right)g \qquad , x_2 = x_1 \\ &\ddot{x}_2 = \ddot{x}_1 \end{aligned} \qquad (4.100)$$

The contact is possible as long as the contact force remains positive.

$$F_{contact} = -\frac{bm}{M+m}\left(\dot{x}_1 - a\Omega\cos\Omega t\right) - \frac{Cm}{M+m}\left(x_1 - a\sin\Omega t\right) \qquad (4.101)$$

It is sensible to introduce the coordinates corresponding to the position of the center of mass of the system and to the distance between the free mass and the base:

$$s = \frac{Mx_1 + mx_2}{M+m};\ r = x_2 - x_1 \qquad (4.102)$$

The new variables are governed by the following equations:

$$\begin{aligned} &s'' = -(1-\mu)\left(s - \mu r - a\sin\omega\tau\right) \\ &\qquad -2\beta(1-\mu)\left(s' - \mu r' - a\omega\cos\omega\tau\right) - p \\ &r'' = \begin{cases} \left(s - \mu r - a\sin\omega\tau\right) + 2\beta\left(s' - \mu r' - a\omega\cos\omega\tau\right), & r > 0 \\ 0, & r = 0 \end{cases} \end{aligned} \qquad (4.103)$$

These equations must be supplemented by the corresponding conditions for the separation of the contact:

$$r = 0 : s = a\sin\omega\tau - 2\beta\left(s' - a\omega\cos\omega\tau\right);\ r' = 0 \qquad (4.104)$$

The following notation is used in these equations:

$$\begin{aligned} &\tau = kt;\ k = \sqrt{\frac{C}{M}};\ \mu = \frac{m}{M+m};\ \omega = \frac{\Omega}{k};\ p = \frac{g}{k^2};\ \beta = \frac{b}{2\sqrt{CM}} \\ &\left(\ \right)' = \frac{d\left(\ \right)}{d\tau} \end{aligned} \qquad (4.105)$$

We assume μ, β and a/p to be small parameters of the same order, neglect the second order small terms with respect to these parameters in the equations and the first order small terms in the separation conditions. Besides that we eliminate the static displacement introducing the new variable

$$s = z - p \tag{4.106}$$

Finally we obtain the equations which have to be analyzed:

$$\begin{aligned} &z'' + z = \mu\left(z - p + r\right) - 2\beta z' + a\sin\omega\tau \\ &r'' = \begin{cases} z - p - \mu r - a\sin\omega\tau + 2\beta z', & r > 0 \\ 0, & r = 0 \end{cases} \end{aligned} \tag{4.107}$$

The separation conditions must be considered alongside the equations (4.107):

$$r = 0: \quad z = p; r' = 0 \tag{4.108}$$

4.5.2. The Master and the Slave Variables; the Unperturbed Solution

Consider the unperturbed system:

$$\begin{aligned} &z_0'' + z_0 = 0 \\ &r_0'' = \begin{cases} z_0 - p, & r_0 > 0 \\ 0, & r_0 = 0 \end{cases} \\ &r_0 = 0: z_0 = p, \; r_0' = 0 \end{aligned} \tag{4.109}$$

The first equation here describes the master variable. The second one together with the separation condition (the last line in the equations (4.109)) describes the slave variable. Our objective now is to build the unperturbed solution with long contacts for the slave variable. The analysis is limited to the basic regime with one collision per one oscillation's period of the base.

The master motion is obvious:

$$z_0 = A\sin\psi; z_0' = A\cos\psi; \; A = const; \; \psi = \tau + const \tag{4.110}$$

In order to obtain the slave solution we have to investigate the free motion $r_0 > 0$ only. It has to satisfy the initial conditions given in the last line of the system (4.109). The result is:

$$r_0'(A,\psi) = A(\cos\psi_0 - \cos\psi) + A(\psi_0 - \psi)\sin\psi_0$$
$$r_0(A,\psi) = A(\sin\psi_0 - \sin\psi) + A(\psi - \psi_0)\cos\psi_0$$
$$-\frac{1}{2}A(\psi - \psi_0)^2 \sin\psi_0 \qquad (4.111)$$
$$\psi_0 = \arcsin\left(\frac{p}{A}\right)$$

This solution exists only if $A > p$. Otherwise the mass wouldn't loose the contact with the base. Solution (4.111) is valid as long as r_0 is positive. This interval can be determined as follows:

$$\psi_0 < \psi < \alpha < 2\pi + \psi_0;$$
$$r_0\big|_{\psi=\alpha} = 0 \Rightarrow \qquad (4.112)$$
$$\frac{1}{2}(\alpha - \psi_0)^2 \sin\psi_0 = (\alpha - \psi_0)\cos\psi_0 + \sin\alpha - \sin\psi_0$$

α is the first solution of the equation (4.112) in the considered interval.

The function r_0 is equal to zero in the interval $\alpha < \psi < \psi_0 + 2\pi$. Then it repeats periodically. One period of this function is shown in Fig. 4.22.

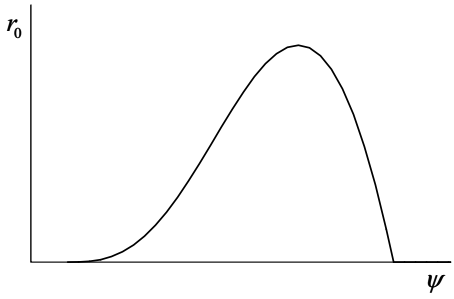

Fig. 4.22. The slave variable r_0

This solution can be used now in order to investigate the resonance of the master subsystem. We can simply replace the slave variable r in the first equation (4.107) through the function r_0 and perform the perturbation analysis as usual. We consider (4.110) as a transformation. The new variables are governed by the following equations:

$$A' = \mu\left(A\sin\psi - p + r_0\right)\cos\psi - 2\beta A\cos^2\psi + a\sin\chi\cos\psi$$
$$\psi' = 1 - \frac{\mu}{A}\left(A\sin\psi - p + r_0\right)\sin\psi + \beta\sin 2\psi - \frac{a}{A}\sin\chi\sin\psi \qquad (4.113)$$
$$\chi' = \omega$$

The frequency delay and the phase difference have to be introduced in order to investigate the main resonance:

$$\delta = \omega - 1;\ \theta = \chi - \psi \qquad (4.114)$$

Finally we arrive at the equations which can be averaged:

$$\begin{aligned} A' &= \mu\left(A\sin\psi - p + r_0\right)\cos\psi - 2\beta A\cos^2\psi \\ &\quad + a\sin\left(\psi + \theta\right)\cos\psi \\ \theta' &= \delta + \left(\mu/A\right)\left(A\sin\psi - p + r_0\right)\sin\psi - \beta\sin 2\psi \\ &\quad + \left(\mu/A\right)\sin\left(\psi + \theta\right)\sin\psi \\ \psi' &= 1 - \left(\mu/A\right)\left(A\sin\psi - p + r_0\right)\sin\psi + \beta\sin 2\psi \\ &\quad - \left(a/A\right)\sin\chi\sin\psi \end{aligned} \qquad (4.115)$$

4.5.3. Equations of the First Order Approximation. Discussion of the Results

Equations of the first order approximation can be obtained by direct averaging of the equations (4.115). The result is:

$$A_1' = \frac{1}{2}a\sin\theta_1 - \beta A_1 + \mu\left\langle r_0\cos\psi\right\rangle$$
$$\theta_1' = \delta + \frac{\mu}{2} + \frac{a}{2A_1}\cos\theta_1 + \frac{\mu}{A_1}\left\langle r_0\sin\psi\right\rangle \qquad (4.116)$$

It is not very difficult to calculate the averages representing the influence of the slave subsystem on the master one.

$$\langle r_0 \cos\psi \rangle = \frac{1}{2\pi} \int_{\psi_0}^{\alpha} r_0 (A_1, \psi) \cos\psi \, d\psi$$

$$= \frac{A_1}{2\pi} \left\{ (\alpha - \psi_0) \sin(\alpha - \psi_0) - \frac{1}{2} (\alpha - \psi_0)^2 \sin\alpha \sin\psi_0 \right. \tag{4.117}$$

$$\left. - \frac{1}{2} (\sin\alpha - \sin\psi_0)^2 + \cos(\alpha - \psi_0) - 1 \right\}$$

$$\langle r_0 \sin\psi \rangle = \frac{1}{2\pi} \int_{\psi_0}^{\alpha} r_0 (A_1, \psi) \sin\psi \, d\psi$$

$$= \frac{A_1}{2\pi} \left\{ \frac{1}{2} (\alpha - \psi_0)^2 \cos\alpha \sin\psi_0 - 2 \sin\psi_0 (\cos\alpha - \cos\psi_0) \right. \tag{4.118}$$

$$\left. - \frac{1}{2} (\alpha - \psi_0) - (\alpha - \psi_0) \cos(\alpha - \psi_0) + \frac{1}{2} (\sin 2\alpha - \sin 2\psi_0) \right\}$$

Setting the right hand sides of the equations (4.116) to zero we obtain a system of transcendental equations governing the stationary resonant regime. This system can be reduced to the following equation for the amplitude:

$$\left(\beta A_1 - \mu \langle r_0 \cos\psi \rangle \right)^2 + \left(\left(\delta + \frac{\mu}{2} \right) A_1 + \mu \langle r_0 \sin\psi \rangle \right)^2 = \frac{a^2}{4} \tag{4.119}$$

Let us investigate which conditions are necessary for the existence of the considered regime. The first one was already mentioned. It guarantees that the amplitude of the base is sufficiently large to allow the separation of the free mass:

$$A_1 > p \tag{4.120}$$

The second one makes the long contact possible (see Fig. 4.23):

$$\alpha > \pi - \psi_0 \tag{4.121}$$

The last condition ensures that there is one long contact in each oscillation's period:

$$\alpha < 2\pi + \psi_0 \tag{4.122}$$

The boundary of the inequality (4.120) means the solution with permanent contact, i.e.

$$r_0 = 0 \Rightarrow A = p = \frac{a}{2\sqrt{\beta^2 + \left(\delta + \frac{\mu}{2}\right)^2}} \tag{4.123}$$

This condition determines the lowest possible amplitude of the excitation:

$$\frac{a}{2p\mu} > \sqrt{\left(\frac{\beta}{\mu}\right)^2 + \left(\frac{\delta}{\mu} + \frac{1}{2}\right)^2} \tag{4.124}$$

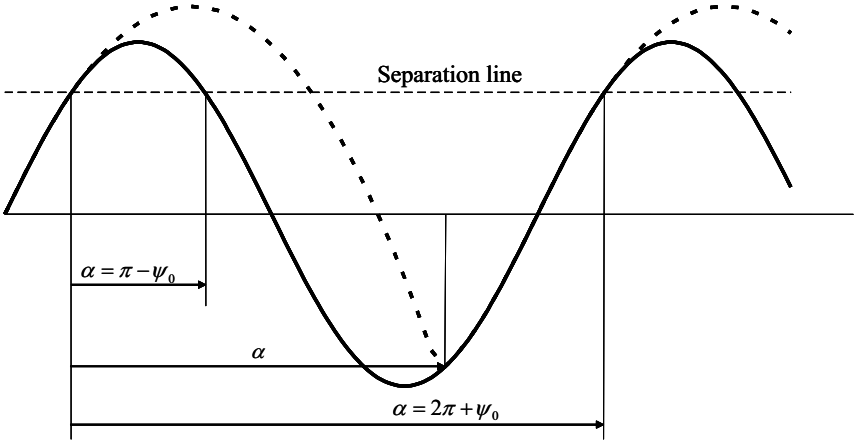

Fig. 4.23. The necessary conditions for the existence of the basic regime

The inequality (4.122) can also be easily solved. It corresponds to the oscillation's amplitude, which can be calculated from (4.112):

$$A_1 = p\sqrt{\pi^2 + 1} \tag{4.125}$$

It is the maximal possible amplitude for the regime with collisions in each oscillation's period. The corresponding condition determines the maximal possible amplitude of the excitation:

$$\frac{a}{2p\mu} < \sqrt{\pi^2 + 1}\sqrt{\left(\frac{\beta}{\mu} + \frac{\pi}{\pi^2 + 1}\right)^2 + \left(\frac{\delta}{\mu} + \frac{1}{2} + \frac{\pi^2 - 1}{\pi^2 + 1}\right)^2} \tag{4.126}$$

The following transcendental equation must be solved in order to determine the boundary corresponding to the inequality (4.121):

$$\psi_0 = \frac{\pi}{2} - \cos\psi_0 \tag{4.127}$$

It can be easily done numerically. This boundary also limits the excitation's amplitude from the bottom. An example of the existence area is shown in Fig. 4.24.

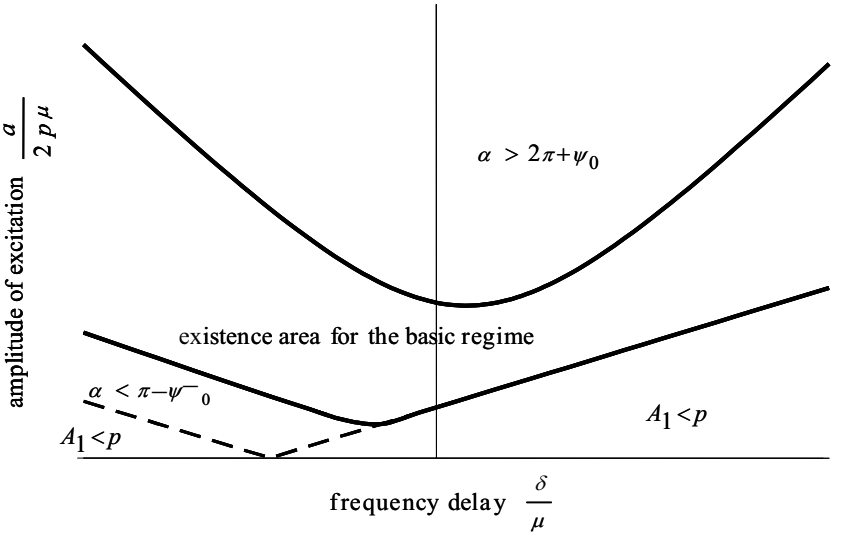

Fig. 4.24. The existence area for the investigated regime with one collision per oscillation's period

4.6. Conclusions

Each of the systems considered in this chapter can be split in two qualitatively different weakly coupled subsystems. The first one is weakly damped, i.e. it is almost conservative. This subsystem can be described as the "master subsystem". The second one is damped strongly and is called the "slave subsystem". The slave subsystem doesn't have its own dynamics. It is determined to some degree through the motions of the master subsystem.

The suggested approach uses this peculiarity. The analysis can be split into three steps. Firstly the general solution to the unperturbed master subsystem has to be found. The corresponding particular solution to the slave subsystem must be found

secondly. Finally this solution can be used in the perturbation analysis for the master system in order to determine the "arbitrary constants" of the general solution.

Two types of strongly damped subsystems were investigated. The approximate results are valid in the case of the strong linear damping after a finite time interval corresponding to transient motions in the strongly damped subsystem.

The situation is more complex for systems with inelastic collisions. Different types of particular solutions are possible here, depending on the parameters of master motions. The described approach is effective if the analysis can be sensibly limited to a particular regime in the dissipative subsystem (for example to fix the number and type of collisions in each oscillation's period of the master subsystem). This is possible in many practical cases dealing with stationary working vibrating machines.

The performed analysis can be used not only in order to investigate the periodic regimes. It is also useful in analyzing of slow transient processes in the master subsystem as long as the chosen regime of the slave subsystem remains unaltered.

The same ideas can be very effective in the analysis of stick-slip oscillations in systems with dry friction. Sticking belongs to the same type of discontinuity as the inelastic impacts do. The degrees of freedom corresponding to the relative motion between the sticking objects can be considered as the slave ones. The corresponding examples don't require any additional ideas and don't contribute significantly to better understanding of the effects or methods.

5. Short Notes on the Significantly Nonlinear Resonance

Significantly nonlinear resonance is an extremely wide, important and complex research field both from practical and theoretical point of view. We have already seen an example of the nonlinear resonance in Chapter 3 as we have analyzed the harmonically excited oscillator in a clearance. But it is not the most important example. We would like to distinguish between the almost linear resonances, which were often investigated above, and the significantly nonlinear resonance (or simply "nonlinear resonance"), which is the object of our interest in this chapter.

The nonlinear resonance occurs always, when the frequency of excitation cannot be considered as a constant, but it depends on the motion of the excited system. This is the case in all applications, where the power of the exciter is comparable with the energy demand of the machine, i.e. all real machines otherwise their drive would be too powerful and expensive. So the practical importance of the nonlinear resonance can be hardly overestimated.

Nonlinear resonance is an extremely complex field from the theoretical point of view. It attracts attention of many different specialists both in mathematics and mechanics for at least 50 years [7, 8, 31, 44, 61, 82 – 84, 86, 87, 113, 114, 138, 139]. Unfortunately mathematical problems in this area are so complex, that a complete theory of the nonlinear resonance still doesn't exist. The main point in the analysis is the applicability of asymptotic expansions for a long time interval $O\left(1/\varepsilon\right)$, despite the natural time scale and the corresponding small parameter is $\sqrt{\varepsilon}$.

Two types of problems are interesting both from practical and theoretical points of view. The first one is the so called passage through the resonance. This problem is general for all machines operating in the overcritical area, i.e. with rotation frequencies higher that the first natural frequency of the oscillating mechanical subsystem. The name "passage through the resonance" itself implies that there should be another possibility. The alternative is called "locking into the resonance" and can be easily explained if we consider the so called "Sommerfeld's effect", which was described firstly by A. Sommerfeld in 1902 and explained by I.Blekhman [17, 20] 50 years later. Actually it was Blekhman, who has introduced the name "Sommerfeld's effect" in 1953.

Consider the system in Fig. 5.1. It consists of an unbalanced rotor mounted on a simple linear oscillating system. In the simplest case the system (the base) has only one degree of freedom. The rotor is driven by a motor with limited power, for

example by an induction motor. Let us increase the applied torque monotonously. The objective is to increase the rotation speed of the rotor. If the base is rigid and fixed, it would be the case. But in the situation under consideration the base has one degree of freedom and the corresponding natural frequency. In this situation the rotation speed of the rotor will increase until it reaches the vicinity of the natural frequency of the base. If we continue to increase the applied torque the following phenomenon can be observed. The rotation speed jumps to the natural frequency of the oscillating system (actually it jumps above this frequency and then after some oscillations stabilizes at the resonance level). The further increase of the rotation speed is going on very slowly (or completely stops), even though the supplied power grows. Instead of that the oscillations of the base increase dramatically. The system remains for a long time in the resonance area. At a certain level of the supplied power the oscillations' amplitude falls down abruptly and the rotation speed jumps to a certain post resonant value.

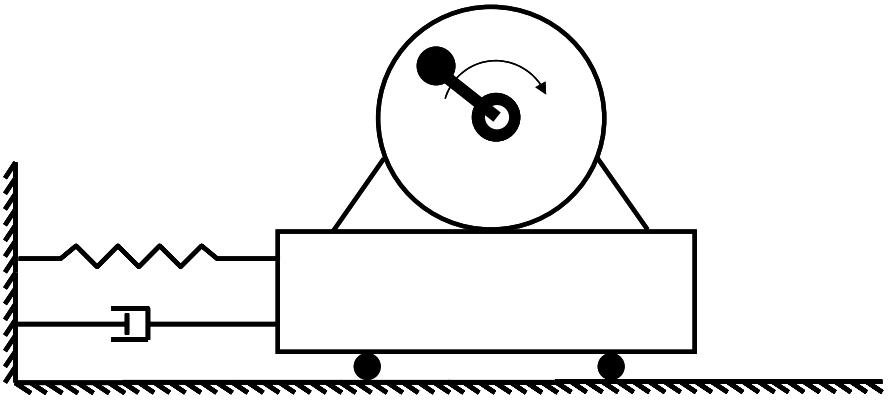

Fig.5.1. An example of the "Sommerfeld's effect"

The described scenario (cf. Fig. 5.2) means that the system finally passes through the resonance. If the supplied power is not sufficient, the system can even remain in the resonance area, i.e. it can be locked by the resonance. This situation can be dangerous for any machine if it is not designed for operating in the resonance. Thus the locking into resonance or too slow passing through the resonance should be avoided.

If we decrease the applied torque and the desired rotation speed from some overcritical level, the effect is even stronger. So to stop a machine may be more dangerous than to start it. The start up and slow down of the induction motor mounted on the system with one degree of freedom is shown in Fig. 5.3 (the simulations were performed here with the higher damping in comparison to the simulations shown in the Fig. 5.2).

The second kind of problems connected with the nonlinear resonance is just the opposite of the first one. For many machines using vibrations for technological purposes operating in the resonance can be desired. Than the problem occurs how to bring a machine to the stable resonance or how to choose the control strategy in order to create a stable and robust resonant regime.

In this chapter we are not going to discuss the whole variety of mathematical and technical problems connected with the nonlinear resonance. It would blow up the limits of this book. We will only briefly discuss one approach useful for the analysis of the desired resonance and illustrate it by two examples from ore processing industry (resonant crushers). Our objective is not to give a complete solution for any particular problem, but to introduce an approach, which is unfortunately less known in mechanical engineering (even though it is well known among the mathematicians).

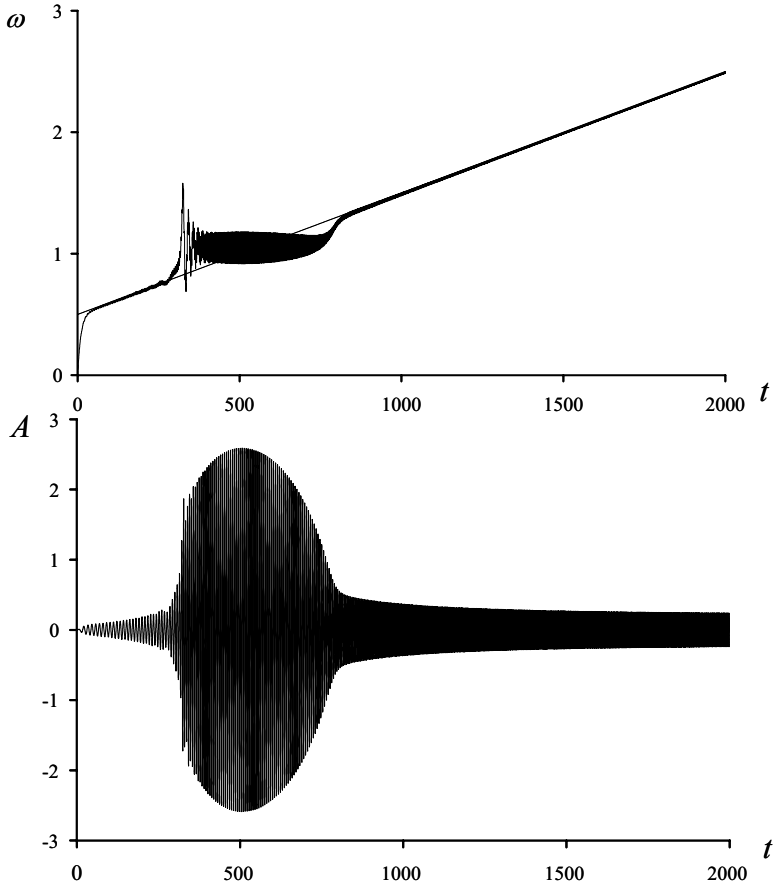

Fig. 5.2. Induction motor passing through the resonance; the figure on the top shows the rotor's speed, the thin line corresponds to the expected velocity of the rotor, the thick line displays its real velocity; the figure at the bottom shows oscillations of the base; all the units are non-dimensional and correspond to the equations (5.1); one can easily notice a significant deviation between the required and achieved rotation speedy coinciding with the strong oscillations of the base

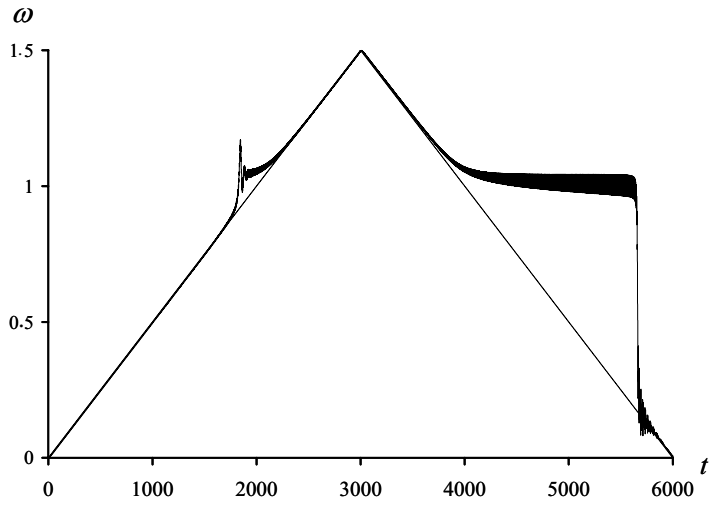

Fig. 5.3. Start up and slow down of an induction motor; the thin line displays the required rotation speed, the thick line shows the really achieved rotation speed; one can notice that the nonlinear behavior is much stronger during the deceleration of the rotor

5.1 The Basic Example of the Nonlinear Resonance

5.1.1 Elementary Analysis and Natural Scale for the Resonance Domain

The unbalanced machine shown in Fig. 5.1 can be considered as the elementary basic system for studying the nonlinear resonance. Its equations of motion in the undimensioned form are as follows (cf. [20]):

$$\begin{aligned} x'' + \beta x' + x &= \varepsilon\left(\psi'' \sin\psi + \psi'^2 \cos\psi\right) \\ \psi'' &= u - v\psi' + \varepsilon k x'' \sin\psi \end{aligned} \tag{5.1}$$

Here x is the coordinate of the base, ψ is the angle of rotation of the rotor, ε is the small parameter proportional to the eccentricity of the rotor and k is a coefficient depending on the inertial properties of the system. The term $\beta x'$ describes the small linear damping in the spring. The term $u - v\psi'$ is the simplest possible description of the induction motor. It takes into account that the motor torque de-

creases with increasing rotation speed; $\omega_0 = u/v$ is the synchronous angular velocity of the rotor, u is the static torque, v is the slope of the motor characteristics.

We assume that ε, β, u and v are the small parameters of the same order, $k = O(1)$. IN comparison to [20] we neglect the gravity forces acting on the unbalanced rotor.

The equations (5.1) can be easily solved with respect to the highest derivatives (we neglect the terms $o(\varepsilon)$):

$$\begin{aligned} x'' + x &= -\beta x' + \varepsilon \psi'^2 \cos\psi \\ \psi'' &= u - v\psi' - \varepsilon k x \sin\psi \end{aligned} \tag{5.2}$$

The corresponding unperturbed system is

$$\begin{aligned} x_0'' + x_0 &= 0 \\ \psi_0'' &= 0 \end{aligned} \tag{5.3}$$

Its general solution can be used as the transformation:

$$\begin{aligned} x &= A\sin\varphi,\ x' = A\cos\varphi \\ \psi' &= \omega \end{aligned} \tag{5.4}$$

Applying (5.4) to (5.2) we obtain:

$$\begin{aligned} A' &= -\beta A\cos^2\varphi + \varepsilon\omega^2 \cos\varphi\cos\psi \\ \omega' &= u - v\omega - \varepsilon k A \sin\varphi\sin\psi \\ \psi' &= \omega \\ \varphi' &= 1 + \beta\sin\varphi\cos\varphi - \varepsilon\frac{\omega^2}{A}\sin\varphi\cos\psi \end{aligned} \tag{5.5}$$

This system has the characteristic form for the nonlinear resonant problems. It has two slow variables A and ω, and two fast rotating phases ψ and φ (all the right hand sides of our equations depend periodically on these variables). If ω would be constant the situation would be simple. The only possible resonance would correspond to

$$\omega_{res} = 1 \tag{5.6}$$

We could introduce the frequency delay $\omega - 1$ as the new small parameter, the phase difference $\psi - \varphi$ as the new variable and investigate the solutions in the

ε -vicinity of the resonance after averaging of all the equations with respect to the only fast rotating phase φ.

Unfortunately, the rotation speed of the rotor is not constant in our case. It is one of the unknown functions. The same statement is valid for the frequency delay $\omega - 1$.

The natural approach transforming (5.5) to the standard form in the vicinity of the resonance (5.6) is based on the idea, that the resonance domain is mach larger in the nonlinear case than in the almost linear case. Let us introduce the frequency delay and the phase difference as follows:

$$\theta = \psi - \varphi,\ \sigma = \frac{\omega - 1}{\sqrt{\varepsilon}} \tag{5.7}$$

The last assumption means that we are going to investigate solutions to our system in the $\sqrt{\varepsilon}$ -vicinity of the resonance (which is actually much larger that the ε -vicinity normally analyzed in the quasi-linear case).

The new variables are governed by the following equations (here we as usual neglect all the terms $o(\varepsilon)$):

$$\begin{aligned}
A' &= -\beta A \cos^2\varphi + \varepsilon \cos\varphi \cos(\varphi + \theta) \\
\sigma' &= \frac{u - v}{\sqrt{\varepsilon}} - \sqrt{\varepsilon} kA \sin\varphi \sin(\varphi + \theta) - v\sigma \\
\theta' &= \sqrt{\varepsilon}\sigma - \beta \sin\varphi \cos\varphi + \frac{\varepsilon}{A} \sin\varphi \cos(\varphi + \theta) \\
\varphi' &= 1 + \beta \sin\varphi \cos\varphi - \frac{\varepsilon}{A} \sin\varphi \cos(\varphi + \theta)
\end{aligned} \tag{5.8}$$

Notice that the coefficient $(u - v)/\sqrt{\varepsilon}$ has the magnitude order $O(\sqrt{\varepsilon})$, so we can denote $(u - v)/\sqrt{\varepsilon} = u_0\sqrt{\varepsilon}$.

This system now has one slow variable $A : A' = O(\varepsilon)$, two semi-slow variables $\sigma : \sigma' = O\sqrt{\varepsilon}$, $\theta : \theta' = O\sqrt{\varepsilon}$ and one fast rotating phase φ. It is a system in the standard form for averaging but with the small parameter $\sqrt{\varepsilon}$ instead of ε. It can be averaged with respect to φ and the result is sufficient for the analysis of the stationary solutions (singular points).

The price we have paid for this result is not small. The averaged equations to (5.8) are valid on the time scale $O\left(\frac{1}{\sqrt{\varepsilon}}\right)$, which is much shorter than in the quasi-linear case. We will see that this time scale is too short for the investigation of transient processes in the vicinity of the nonlinear resonance!

The result of the averaging of the equations (5.8) is as follows (we use the second order approximation with respect to the new small parameter $\sqrt{\varepsilon}$, which has the accuracy $O(\varepsilon)$):

$$\begin{aligned} A_2' &= -\frac{1}{2}\beta A_2 + \frac{1}{2}\varepsilon\cos\theta_2 \\ \sigma_2' &= u_0\sqrt{\varepsilon} - \frac{1}{2}\sqrt{\varepsilon}kA_2\cos\theta_2 - \nu\sigma_2 \\ \theta_2' &= \sqrt{\varepsilon}\sigma_2 - \frac{\varepsilon}{2A_2}\sin\theta_2 \end{aligned} \tag{5.9}$$

The subscript 2 denotes here the averaged variables in the second order approximation. In order to analyze this system let us consider the first order approximation (we neglect the terms $o\left(\sqrt{\varepsilon}\right)$):

$$\begin{aligned} A_1' &= 0 \\ \sigma_1' &= u_0\sqrt{\varepsilon} - \frac{1}{2}\sqrt{\varepsilon}kA_1\cos\theta_1 \\ \theta_1' &= \sqrt{\varepsilon}\sigma_1 \end{aligned} \tag{5.10}$$

The amplitude A_1 remains constant according to the first equation. The second and the third equations together describe the so called "*equivalent pendulum*".

$$\theta_1'' + \frac{1}{2}\varepsilon kA_1\cos\theta_1 = \varepsilon u_0 \tag{5.11}$$

5.1.2 The Basic Regimes of the Equivalent Pendulum

The equation (5.11) describes a mathematical pendulum with an applied constant torque. We have already seen this equation in section 3.6. The small formal difference can be eliminated if we change the zero of the phase difference θ:

$$\theta_1 = \vartheta_1 + \frac{\pi}{2}$$
$$\vartheta_1'' - \frac{1}{2}\varepsilon k A_1 \sin\vartheta_1 = \varepsilon u_0 \tag{5.12}$$

The phase portrait of the equivalent pendulum for the case $|kA_1| > |2u_0|$ is shown in Fig. 5.4.

The first and very important property of the system (5.12) is its conservativeness. It is the general property of the first order approximation to all systems with two rotating phases in the vicinity of a simple nonlinear resonance.

The term $\frac{1}{2}\varepsilon k A_1 \sin\vartheta$ is usually called "*the vibrational torque*". Its relation to the "*external torque*" εu_0 determines the basic properties of the equivalent pendulum. Two types of motion are possible for the pendulum depending on the parameter values and its initial energy: rotation and oscillations. Only rotation is possible if the external torque is sufficient large:

$$u_0 > kA_1/2 \tag{5.13}$$

The stationary resonance is impossible in that case. So the resonance is not attracting and the system leaves its vicinity on its natural time scale $t = O\left(\sqrt{\varepsilon}\right)$.

The situation changes, if the external torque is small, i.e.

$$u_0 < kA_1/2 \tag{5.14}$$

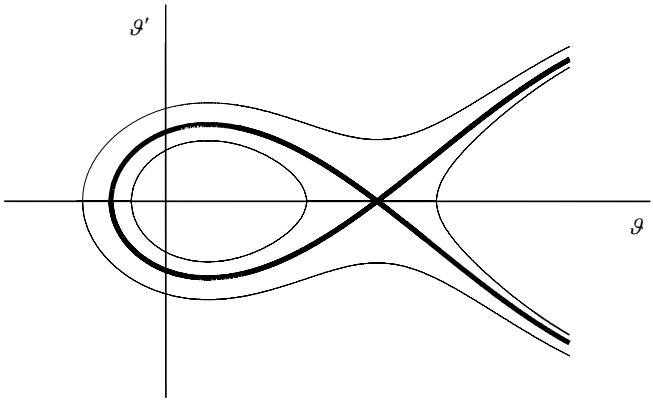

Fig. 5.4. The phase portrait of the equivalent resonant pendulum; the homoclinic loop (thick line) limits the attraction area of the stationary solution (according to the first order approximation).

Then the closed loop going through the saddle point exists. It separates the area of oscillations (inside of the loop) from the area of rotations (outside of the loop). The type of motion depends on the initial conditions, especially on the energy of the equivalent pendulum. The total energy of the pendulum can be obtained as the first integral to the equation (5.12):

$$E = \tfrac{1}{2}\vartheta_1'^2 + \tfrac{1}{2}\varepsilon k A_1 \cos\vartheta_1 - \varepsilon u_0 \vartheta_1 \tag{5.15}$$

The first term here is the kinetic energy E_K, the second and the third terms correspond to the potential energy E_P of the equivalent pendulum (see Fig. 5.5).

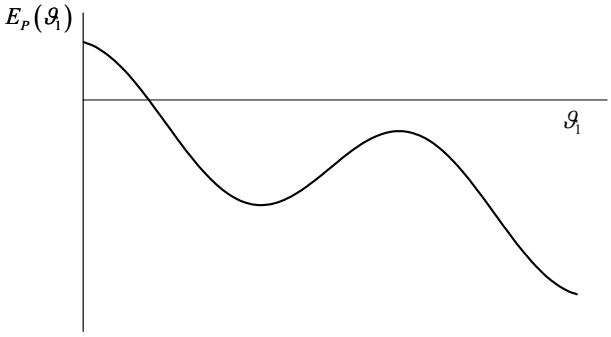

Fig. 5.5. Potential energy of the equivalent pendulum

If the initial energy of the pendulum is between the maximum and the minimum of one potential well, then the pendulum oscillates in it. If the initial energy is larger than the maximum the pendulum rotates with increasing speed.

Unfortunately the system of the first order approximation is conservative. It means we cannot conclude anything about the stability of the resonant solutions on its basis. The second order approximation however gives us sufficient information in order to investigate the stationary resonant regime.

5.1.3 Stability of the Stationary Resonance

The stability investigation based on the system (5.9) can be performed for the stationary resonant solutions. Let us convert to variable $\vartheta = \theta - \dfrac{\pi}{2}$ in the equations (5.9):

$$
\begin{aligned}
A_2' &= -\frac{1}{2}\beta A_2 - \frac{1}{2}\varepsilon \sin\vartheta_2 \\
\sigma_2' &= u_0\sqrt{\varepsilon} + \frac{1}{2}\sqrt{\varepsilon}kA_2 \sin\vartheta_2 - v\sigma_2 \\
\vartheta_2' &= \sqrt{\varepsilon}\sigma_2 - \frac{\varepsilon}{2A_2}\cos\vartheta_2
\end{aligned}
\tag{5.16}
$$

The stationary resonance satisfies the following equations:

$$
\begin{aligned}
A_{20} &= -\frac{1}{\beta_0}\sin\vartheta_{20} \\
\sin\vartheta_{20} &= -\frac{2u_0}{kA_{20}} + \frac{2v_0}{kA_{20}}\sqrt{\varepsilon}\sigma_{20} \\
\sigma_{20} &= \frac{\sqrt{\varepsilon}}{2A_{20}}\cos\vartheta_{20}
\end{aligned}
\tag{5.17}
$$

Here we have introduced the following notation: $\beta = \beta_0\varepsilon,\ v = v_0\varepsilon$ in order to compare the magnitude orders explicitly.

It follows from the last equation in (5.17) that $\sigma_{20} = O\left(\sqrt{\varepsilon}\right) \approx 0$. Hence $\sin\vartheta_{20} = -\dfrac{2u_0}{kA_{20}} + O(\varepsilon)$. Substituting this relationship into the first equation we obtain the stationary amplitude of the resonator (here we neglect the small terms):

$$
A_{20} = \sqrt{\frac{2u_0}{k\beta_0}} \tag{5.18}
$$

Two different stationary regimes of the equivalent pendulum can correspond to this amplitude:

$$
\begin{aligned}
&\sin\vartheta_{20} = -\frac{2u_0}{kA_{20}} \Rightarrow \\
&\vartheta_{20}^{(1)} = -\arcsin\left(\sqrt{\frac{2u_0\beta_0}{k}}\right);\ \vartheta_{20}^{(2)} = \pi + \arcsin\left(\sqrt{\frac{2u_0\beta_0}{k}}\right)
\end{aligned}
\tag{5.19}
$$

Referring back to the relationship for the potential energy of the equivalent pendulum one can easily notice that its minimum corresponds to the stationary solution with $\cos\vartheta_{20} < 0$ and its maximum corresponds to the solution with $\cos\vartheta_{20} > 0$. Thus the first solution is obviously unstable and only the second one can be stable.

In order to investigate the stability of the second solution let us write the equations in variations:

$$
\begin{aligned}
&A_2 = A_{20} + \tilde{A};\ \vartheta_2 = \vartheta_{20} + \tilde{\vartheta};\ \sigma_2 = \sigma_{20} + \tilde{\sigma} \\
&\tilde{A}' = -\left(\frac{1}{2}\beta\right)\tilde{A} - \left(\frac{1}{2}\varepsilon\cos\vartheta_{20}\right)\tilde{\vartheta} \\
&\tilde{\sigma}' = \left(\frac{1}{2}\sqrt{\varepsilon}k\sin\vartheta_{20}\right)\tilde{A} + \left(\frac{1}{2}\sqrt{\varepsilon}kA_{20}\cos\vartheta_{20}\right)\tilde{\vartheta} - v\tilde{\sigma} \\
&\tilde{\vartheta}' = \sqrt{\varepsilon}\tilde{\sigma} + \left(\frac{\varepsilon}{2A_{20}}\sin\vartheta_{20}\right)\tilde{\vartheta} + \left(\frac{\varepsilon}{2A_{20}^2}\cos\vartheta_{20}\right)\tilde{A}
\end{aligned}
\tag{5.20}
$$

The corresponding characteristic equation for the eigenvalues λ is:

$$
\begin{vmatrix}
\frac{1}{2}\beta + \lambda & \frac{1}{2}\varepsilon\cos\vartheta_{20} & 0 \\
-\frac{\varepsilon}{2A_{20}^2}\cos\vartheta_{20} & -\frac{\varepsilon}{2A_{20}}\sin\vartheta_{20} + \lambda & -\sqrt{\varepsilon} \\
-\frac{1}{2}\sqrt{\varepsilon}k\sin\vartheta_{20} & -\frac{1}{2}\sqrt{\varepsilon}kA_{20}\cos\vartheta_{20} & v + \lambda
\end{vmatrix} = 0
\tag{5.21}
$$

This equation is a polynomial of the third order with respect to λ:

$$
\begin{aligned}
&\lambda^3 + \lambda^2\varepsilon\left(v_0 + \beta_0\right) \\
&-\lambda\left(\frac{1}{2}\varepsilon kA_{20}\cos\vartheta_{20} + \frac{1}{4A_{20}^2}\varepsilon^2\cos^2\vartheta_{20} + \beta_0 v_0\varepsilon^2 + \frac{\varepsilon^2{\beta_0}^2}{4}\right) \\
&+\frac{1}{4}\varepsilon^2 k\cos\vartheta_{20}\sin\vartheta_{20} + \frac{\varepsilon^3{\beta_0}^2 v_0}{4} = 0
\end{aligned}
\tag{5.22}
$$

Here the relationships (5.17) were taken into account. The terms $O\left(\varepsilon^2\right)$ in the third term and $O\left(\varepsilon^3\right)$ can be neglected. This simplifies our equation:

$$\lambda^3 + \lambda^2 \varepsilon \left(v_0 + \beta_0 \right) - \lambda \frac{\varepsilon k}{2} A_{20} \cos \vartheta_{20} - \frac{1}{4} \varepsilon^2 k \beta_0 A_{20} \cos \vartheta_{20} = 0 \qquad (5.23)$$

The Hurwitz' criterion for the asymptotic stability (the sufficient condition that all the eigenvalues have negative real parts) is:

$$\begin{aligned} &\cos \vartheta_{20} < 0 \\ &-\frac{\varepsilon^2 k}{2} \left(v_0 + \beta_0 \right) A_{20} \cos \vartheta_{20} > -\frac{1}{4} \varepsilon^2 k \beta_0 A_{20} \cos \vartheta_{20} \end{aligned} \qquad (5.24)$$

The first inequality is already known. It means that only the equilibrium position corresponding to the minimum of the potential energy can be stable. The second one is fulfilled automatically because it can be transformed as follows:

$$v_0 + \frac{1}{2} \beta_0 > 0 \qquad (5.25)$$

Both terms in (5.25) are positive due to their physical sense: β is the damping in the mechanical part of the system and "$-v$" is the negative slope of the motor's torque.

Thus the equilibrium point $A_{20}, \vartheta_{20}^{(2)}, \sigma_{20} = 0$ is stable.

Notice that this result is based on the second order approximation with respect to the natural small parameter $\sqrt{\varepsilon}$. The first order approximation was not sufficient to solve the stability problem.

The basic necessary condition for the existence of the resonance is based on the inequality (5.19):

$$A_{20} > 2u_0 / k \quad \Rightarrow \quad 2u_0 \beta_0 < k \qquad (5.26)$$

5.1.4 Resonant Motions: Averaging with Respect to the Oscillations of the Equivalent Pendulum

(This section contains advanced approaches and can be omitted by readers, who are not interested in the special problems of the nonlinear resonance.)

We have seen in the previous section that the damping of the equivalent pendulum is very small. It is sensible to investigate its behavior carefully in such a situation. Unfortunately the averaged equations (5.16) are valid for the time interval $O\left(1/\sqrt{\varepsilon}\right)$, which is too short for analyzing of the transient processes in the system. The amplitude of the resonant oscillations increases with the rate $O(\varepsilon)$.

Thus on the natural resonant time scale it cannot change sufficiently in order to achieve considerable values: $O(\varepsilon)\cdot O\left(1/\sqrt{\varepsilon}\right)=O\left(\sqrt{\varepsilon}\right)$. What we really need in order to investigate the transient solution (and from the practical point of view in order to control them) is an approximation which would be valid on the time scale $O(1/\varepsilon)$.

Careful analysis shows that the source of the increasing inaccuracy of our approximation is not the amplitude of the mass or the amplitude of the equivalent pendulum, but it is its phase. This result together with the fact that the equivalent pendulum oscillates in the resonance domain, explains the idea to average the equations (5.16) with respect to the phase of the equivalent pendulum's oscillations. This operation is possible because the "unperturbed system" (5.12) is conservative, it can be integrated and its integrals can be considered as the new variables for the full system (5.16). However the technical problems are considerable, because the corresponding transformation cannot be performed using elementary or even widely known special functions. We are going to perform the analysis applying a transformation, which was suggested by A. Pechenv [87], who also proved that the final twice averaged system is valid on the required sufficient long time interval [86]. This proof can be found in the Appendix VII.

It is convenient to choose the zero of the potential energy in the stable equilibrium of the equivalent pendulum for particular amplitude A_2:

$$\begin{aligned} E_P &= \frac{1}{2}\varepsilon k A_2\left(\cos\vartheta_2-\cos\vartheta_*\right)-\varepsilon u_0\left(\vartheta_2-\vartheta_*\right)\\ E_K &= \frac{1}{2}\varepsilon\sigma_2^2\\ \vartheta_* &= \pi+\arcsin\left(\frac{2u_0}{kA_2}\right),\ \sin\vartheta_*=-\frac{2u_0}{kA_2},\ \cos\vartheta_*<0 \end{aligned} \tag{5.27}$$

Now we can introduce new variables g and γ, replacing the variables ϑ and σ:

$$\begin{aligned} &\frac{1}{2}g^2\cos^2\gamma=\frac{1}{2}kA_2\left(\cos\vartheta_2-\cos\vartheta_*\right)-u_0\left(\vartheta_2-\vartheta_*\right)\\ &g\sin\gamma=\sigma_2;\ \operatorname{sgn}\left(\cos\gamma\right)=\operatorname{sgn}\left(\vartheta_*-\vartheta_2\right) \end{aligned} \tag{5.28}$$

Notice that ϑ_* is not a constant. It depends on the variable A_2. The variable g^2 describes the full energy of the pendulum according to the first order approximation (cf. Fig. 5.6).

This transformation enables to express the old variables uniquely in terms of the new ones, but it does not permit analytic solutions of the transcendental equation for ϑ_2 as the reciprocal to (5.28) in elementary functions. It is assumed below that ϑ_2 is expressed in terms of the variables A_2, g, γ.

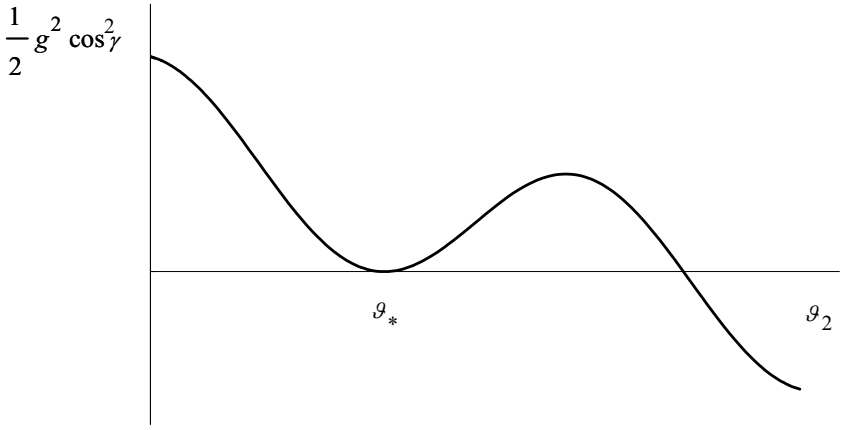

Fig. 5.6. Potential energy of the equivalent pendulum in the new variables

The new variables are governed by the following equations (we neglect the terms $O(\varepsilon)$ in the equation for γ):

$$
\begin{aligned}
A_2' &= -\frac{1}{2}\beta_0\varepsilon A_2 - \frac{1}{2}\varepsilon\sin\vartheta_2 \\
g' &= -\varepsilon\nu_0 g\sin^2\gamma + \varepsilon\frac{k}{4g}\cos\vartheta_2\left(\sin\vartheta_2 - \sin\vartheta_*\right) \\
&\quad -\varepsilon\frac{k}{4g}\left(\sin\vartheta_2 + 2\beta_0 A_2\right)\left(\cos\vartheta_2 - \cos\vartheta_*\right) \\
\gamma' &= \frac{1}{2}\sqrt{\varepsilon}kA_2\frac{\sin\vartheta_2 - \sin\vartheta_*}{g\cos\gamma}
\end{aligned}
\tag{5.29}
$$

Here we suppose that the variable ϑ_2 is expressed in terms of the new variables according to the transformation (5.28). Unfortunately it cannot be done in elementary functions in a closed form. Let us show that γ' is a bounded function. The only zero of the function $g\cos\gamma$ corresponds to the point $\vartheta_2 = \vartheta_*$ (cf. Fig. 5.6). In the vicinity of this point we can solve the equations (5.28) taking the quadratic terms in the corresponding Taylor's expansion into account:

$$g\cos\gamma \approx -\sqrt{-\tfrac{1}{2}\cos\vartheta_*}\left(\vartheta_2 - \vartheta_*\right) \tag{5.30}$$

Applying this relationship we obtain the following estimations:

$$\begin{aligned}\lim_{g\cos\gamma\to 0}\frac{\sin\vartheta_2 - \sin\vartheta_*}{g\cos\gamma} &= -\frac{\cos\vartheta_*}{\sqrt{-\tfrac{1}{2}\cos\vartheta_*}} > 0\\ \lim_{g\cos\gamma\to 0}\frac{\cos\vartheta_2 - \cos\vartheta_*}{g\cos\gamma} &= \frac{\sin\vartheta_*}{\sqrt{-\tfrac{1}{2}\cos\vartheta_*}}\end{aligned} \tag{5.31}$$

Thus the right hand side of the equation for γ' remains always positive and bounded (cf. Fig. 5.7).

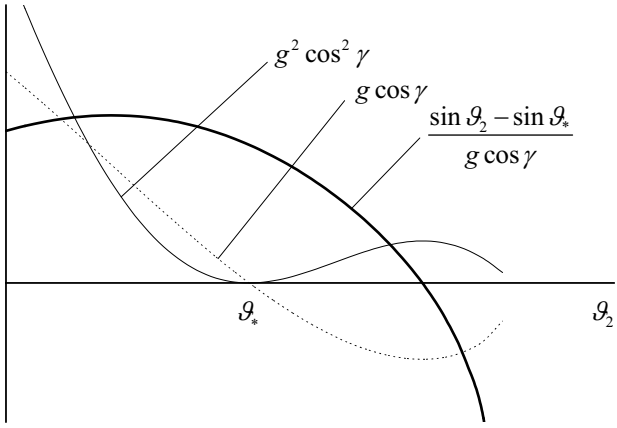

Fig. 5.7. The right hand side of the equation for γ' is bounded and positive in the area of oscillations of the equivalent pendulum; it becomes zero if we approach to the maximum of the potential energy

The variable γ can be interpreted as a rotating phase, which velocity has the magnitude order $O\left(\sqrt{\varepsilon}\right)$. We can convert to it as the independent variable:

$$\frac{dA_2}{d\gamma} = -\sqrt{\varepsilon}\,\frac{\beta_0}{k}\,\frac{g\cos\gamma}{\sin\vartheta_2 - \sin\vartheta_*} - \sqrt{\varepsilon}\,\frac{1}{kA_2}\,\frac{\sin\vartheta_2 g\cos\gamma}{\sin\vartheta_2 - \sin\vartheta_*}$$

$$\frac{dg}{d\gamma} = -\sqrt{\varepsilon}\,\frac{2v_0 g^2\sin^2\gamma\cos\gamma}{kA_2\left(\sin\vartheta_2 - \sin\vartheta_*\right)} + \sqrt{\varepsilon}\,\frac{1}{2A_2}\cos\vartheta_2\cos\gamma \qquad (5.32)$$

$$-\sqrt{\varepsilon}\,\frac{k}{2A_2}\left(\sin\vartheta_2 + 2\beta_0 A_2\right)\cos\gamma\,\frac{\cos\vartheta_2 - \cos\vartheta_*}{\sin\vartheta_2 - \sin\vartheta_*}$$

This system is in the standard form for averaging. The small parameter is $\sqrt{\varepsilon}$. After averaging with respect to the "*semi-slow motions*" in the resonance domain the system can be represented in the plane of two variables A_2 and g^2. Unfortunately no results are available in the closed form. A typical attraction area of the stationary resonance is shown in Fig. 5.8 (cf. [87]).

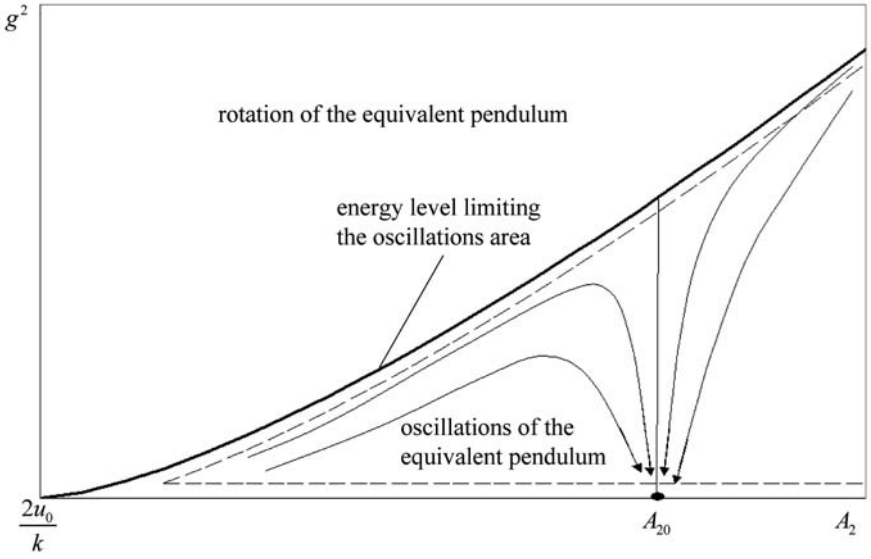

Fig. 5.8. The plane of the averaged variables A_2 and g^2 for sufficiently large negative slope of the engine's characteristics

One can find there the stable equilibrium point $g^2 = 0,\ A_2 = A_{20}$ and the limiting curve corresponding to the closed homoclinic loop in Fig. 5.4. All the solutions under the limiting curve converge to the stable stationary solution. If the slope of the motor's characteristics is small, then a part of the area right from the stationary amplitude must be excluded from the attraction area.

These particular results illustrate the possibility to investigate the attraction area of the nonlinear resonance. Much more interesting is the general fact concerning the validity of the twice averaged equations.

It can be shown that the equations (5.32) averaged with respect to the semi-slow phase are valid with the accuracy of $O\left(\sqrt{\varepsilon}\right)$ for the long time interval $t = O\left(\varepsilon^{-1}\right)$

The phase γ after averaging can be estimated very inaccurately with a large mistake $O(1)$. These results based on the so called hierarchical averaging of systems containing slow, semi-slow and fast variables give mathematical foundation for analyzing of the nonlinear resonance.

5.2 Nonlinear Resonant Crusher with Almost Elastic Collisions

5.2.1 Problem Description. Equations of Motion

Consider the system shown in Fig. 5.9. It consists of a frame of mass M_1 in whose interior a striker of mass m is attached elastically. The striker can collide almost elastically with the frame at a relative distance Δ_f from the static equilibrium point. An inertial vibrator characterized by the mass m_r and radius ρ is placed on the frame and set in rotation by an induction motor. The analysis below is confined to linear approximation of the torque at the motor's shaft:

$$P\left(\dot{\psi}\right) = U - V\dot{\psi}, U > 0, V > 0 \tag{5.33}$$

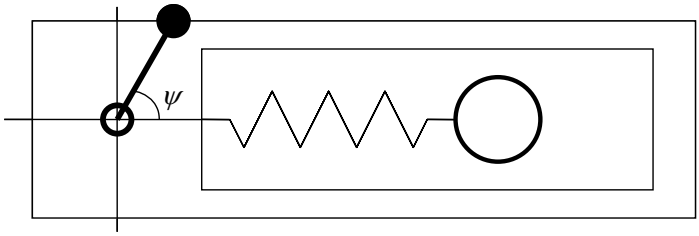

Fig. 5.9. Resonant crusher with almost elastic collisions

The kinetic and potential energies of the system together with the dissipative function can be written as follows:

$$\begin{aligned} T &= \tfrac{1}{2} M\dot{x}_1^2 + m\dot{x}_1\dot{q} + \tfrac{1}{2} m\dot{q}^2 + \tfrac{1}{2} m_r \rho^2 \dot{\psi}^2 - m_r \rho \dot{x}_1 \dot{\psi} \sin\psi \\ \Pi &= \tfrac{1}{2} cq^2; \quad W = \tfrac{1}{2} b\dot{q}^2 \end{aligned} \tag{5.34}$$

Here x_1 is the co-ordinate of the frame, $x_2 = x_1 + q$ is the mass of the striker, c and b are the stiffness and the damping of the spring correspondingly, $M = M_1 + m + m_r$ is the total mass of the system.

Now the equations of motion between the collisions and the kinematical conditions describing collisions must be formulated. For the time intervals between the collisions one obtains:

$$\begin{cases} M\ddot{x}_1 + m\ddot{q} - m_r \rho \left(\ddot{\psi} \sin\psi + \dot{\psi}^2 \cos\psi \right) = 0 \\ m\ddot{x}_1 + m\ddot{q} + b\dot{q} + cq = 0 \\ m_r \rho^2 \ddot{\psi} - m_r \rho \ddot{x}_1 \sin\psi = U - V\dot{\psi} \end{cases}, \quad \text{if} \quad q < \Delta_f \tag{5.35}$$

Now the kinematical conditions for the collisions have to be formulated. The main peculiarity of the system is that the rotor is placed on the colliding frame. It means that not only the velocities of the colliding bodies, but also the rotor's angular velocity is discontinuous at the time point of the collision. The collisions take place when $q = \Delta_f$.

The collisions are described conventionally by the generalized impulse conservation law accomplished by the Newton's impact law (*cf.* Chapter 1):

$$\begin{aligned} & p_{1+} \equiv \left.\frac{\partial T}{\partial \dot{x}_1}\right|_+ = p_{1-} \equiv \left.\frac{\partial T}{\partial \dot{x}_1}\right|_-; \qquad p_{\psi+} \equiv \left.\frac{\partial T}{\partial \dot{\psi}}\right|_+ = p_{\psi-} \equiv \left.\frac{\partial T}{\partial \dot{\psi}}\right|_- \\ & \dot{q}_+ = -R\dot{q}_- \end{aligned} \tag{5.36}$$

These conditions can be formulated explicitly as follows:

$$\begin{aligned} & \dot{x}_{1+} - \dot{x}_{1-} = -\frac{m\left(\dot{q}_+ - \dot{q}_-\right)}{M - m_r \sin^2\psi} \\ & \dot{\psi}_+ - \dot{\psi}_- = -\frac{m\sin\psi\left(\dot{q}_+ - \dot{q}_-\right)}{\rho\left(M - m_r \sin^2\psi\right)}, \qquad \text{if} \quad q = \Delta_f \\ & \dot{q}_+ - \dot{q}_- = -(1+R)\dot{q}_- \end{aligned} \tag{5.37}$$

Equations (5.35) and (5.37) describe motion of the system under consideration completely. It is sensible now to eliminate the variable x_1, to convert to undimensionedl variables and to introduce small parameters.

$$\begin{aligned} &k=\sqrt{\frac{c}{m}\frac{M}{M-m}},\beta=\frac{bM}{km\ \ (M-m)},\mu=\frac{m}{M-m},\varepsilon=\frac{m_r}{M}\\ &\delta=\frac{\Delta}{\rho},u=\frac{U}{m\rho^2k^2},v=\frac{V}{m\rho^2k^2},y=\frac{q}{\rho},\tau=kt,\frac{d(\ )}{d\tau}=(\ )' \end{aligned} \tag{5.38}$$

Now the equations (5.35) and the impact conditions (5.37) can be rewritten as follows:

$$\begin{cases} y''+\beta y'+y=-\varepsilon(1+\mu)\left(\psi''\sin\psi+\psi'^2\cos\psi\right)\\ \psi''=u-v\psi'+\varepsilon\left(\psi''\sin\psi+\psi'^2\cos\psi\right)\\ \qquad +\sin\psi\left\{\beta\mu(1+\mu)y'+\mu(1+\mu)y\right.\\ \qquad \left.+\varepsilon\mu\left(\psi''\sin\psi+\psi'^2\cos\psi\right)\right\} \end{cases},\ y<\delta \tag{5.39}$$

$$\begin{cases} y'_+-y'_-=-(1+R)y'_-\\ \psi'_+-\psi'_-=\dfrac{\mu\sin\psi\,(1+R)}{(1+\mu)\left(1-\varepsilon\sin^2\psi\right)}y'_- \end{cases},\ y=\delta \tag{5.40}$$

We suppose now $\beta,\varepsilon,\mu,\delta,(1-R),u,v$ to be small parameters of the same order and omit the terms $o(\varepsilon)$:

$$\begin{cases} y''+\beta y'+y=-\varepsilon\psi'^2\cos\psi\\ \psi''=u-v\psi'+\varepsilon\psi'^2\sin\psi\cos\psi+\mu y\sin\psi \end{cases},\ y<\delta$$
$$\begin{cases} y'_+-y'_-=-(1+R)y'_-\\ \psi'_+-\psi'_-=\mu\cos\psi\,(1+R)y'_- \end{cases},\ y=\delta \tag{5.41}$$

System (5.41) shows that both variables y and ψ are discontinuous. But the arts of the discontinuity of these variables are significantly different. The jump of the variable y is large. The discontinuity of the variable ψ is on the contrary small. Due to this circumstance different methods can be applied for the averaging of these discontinuities. Variable ψ can be averaged "directly". Variable y must

be regularized by means of the discontinuous unfolding transformation, which was discussed in Chapter 3.

5.2.2 The Unfolding Transformation. The Main Resonance

In order to regularize the variable y the following unfolding variable transformation can be applied:

$$y = \delta - |z| \tag{5.42}$$

The new variable z contains only small discontinuities:

$$\begin{aligned}
&z'' + \beta z' + z = \delta \operatorname{sgn} z + \varepsilon\omega^2 \cos\psi \operatorname{sgn} z \\
&z'_+ - z'_- = -(1-R) z'_-, \text{ if } z = 0 \\
&\psi' = \omega \\
&\omega' = u - v\omega + \varepsilon\omega^2 \sin\psi \cos\psi + \mu(\delta - |z|)\sin\psi \\
&\omega_+ - \omega_- = (1+R)\mu |z'_-| \sin\psi, \quad \text{if } z = 0
\end{aligned} \tag{5.43}$$

In order to obtain a system in standard form the van-der-Pol's transformation based on the solution to the unperturbed system can be applied:

$$z = A\sin\varphi; \qquad z' = A\cos\varphi \tag{5.44}$$

Substituting (5.44) into (5.43) and neglecting the small terms of the second order one obtains:

$$\begin{aligned}
&A' = -\beta A\cos^2\varphi + \delta\cos\varphi \operatorname{sgn}(\sin\varphi) \\
&\qquad\qquad + \varepsilon\omega^2 \cos\psi \cos\varphi \operatorname{sgn}(\sin\varphi) \\
&A_+ - A_- = -(1-R) A_-, \quad \text{if } \varphi = \pi n \\
&\omega' = u - v\omega + \varepsilon\omega^2 \sin\psi\cos\psi - \mu A |\sin\varphi| \sin\psi \\
&\omega_+ - \omega_- = (1+R)\mu A \sin\psi, \quad \text{if } \varphi = \pi n \\
&\psi' = \omega \\
&\varphi' = 1 + \frac{1}{2}\beta\sin 2\varphi - \frac{\delta}{A}|\sin\varphi| - \frac{\varepsilon}{A}\omega^2 \cos\psi\,|\sin\varphi|
\end{aligned} \tag{5.45}$$

The main principal resonance in the system (5.45) corresponds to the surface $\omega = 2$. Two semi slow variables can be introduced in its $\sqrt{\varepsilon}$ -vicinity:

$$\theta = \psi - 2\varphi; \qquad \sigma = \frac{\omega - 2}{\sqrt{\mu}} \tag{5.46}$$

Substituting (5.46) into (5.45) one obtains equations in standard form for averaging:

$$\begin{aligned}
A' &= -\beta A \cos^2\varphi + \delta \cos\varphi \operatorname{sgn}(\sin\varphi) \\
&\quad + \varepsilon\omega^2 \cos(2\varphi+\theta)\cos\varphi \operatorname{sgn}(\sin\varphi) \\
A_+ - A_- &= -(1-R)A_-, \quad \text{if} \quad \varphi = \pi n \\
\sigma' &= \frac{u}{\sqrt{\mu}} - \frac{v}{\sqrt{\mu}}\omega + \frac{\varepsilon}{\sqrt{\mu}}\omega^2 \sin(2\varphi+\theta)\cos(2\varphi+\theta) \\
&\quad - \sqrt{\mu} A |\sin\varphi| \sin(2\varphi+\theta) \\
\sigma_+ - \sigma_- &= (1+R)\sqrt{\mu} A \sin\theta, \quad \text{if} \quad \varphi = \pi n \\
\theta' &= \sqrt{\mu}\sigma - \beta \sin 2\varphi + 2\frac{\delta}{A}|\sin\varphi| + 2\frac{\varepsilon}{A}\omega^2 \cos(2\varphi+\theta)|\sin\varphi| \\
\varphi' &= 1 + \frac{1}{2}\beta \sin 2\varphi - \frac{\delta}{A}|\sin\varphi| - \frac{\varepsilon}{A}\omega^2 \cos(2\varphi+\theta)|\sin\varphi|
\end{aligned} \tag{5.47}$$

These equations can be averaged with respect to the fast rotating phase φ.

5.2.3 Averaging with Respect to the Fast Rotating Phase. Stationary Regimes

The result of the averaging (the second order approximation) is as follows:

$$\begin{aligned}
A_2' &= -\left(\frac{1}{2}\beta + \frac{1-R}{\pi}\right)A_2 - \frac{4}{3\pi}\varepsilon\omega^2 \sin\theta_2 \\
\theta_2' &= \sqrt{\mu}\sigma_2 + \frac{4\delta}{\pi A_2} - \frac{4}{3\pi A_2}\varepsilon\omega^2 \cos\theta_2 \\
\sigma_2' &= \frac{u - v\omega}{\sqrt{\mu}} - v\sigma_2 + \left(\frac{2}{3\pi} + \frac{1+R}{\pi}\right)\sqrt{\mu} A_2 \sin\theta_2
\end{aligned} \tag{5.48}$$

The effective damping β_e can be introduced here similar to Chapter 3.

$$\beta_e = \frac{1}{2}\beta + \frac{1-R}{\pi} \tag{5.49}$$

This effective damping includes both the energy dissipation in the spring between the collisions and energy losses during collisions. But the most interesting effect of impacts in the system (5.48) is not the increased effective damping. The most interesting effect is expressed through the last term in the last equation, which is significantly increased due to the discontinuous condition for σ (the fourth equation in (5.47)). In other words the contribution of the discontinuous jumps in the rotor's rotation speed increases significantly the stiffness of the equivalent pendulum or the vibrational torque [20] in the last equation:

$$\begin{aligned} &\theta_{20}'' - \mu A_{20} W \sin\theta_{20} = \mu u_0 \\ &u_0 = \frac{u - \omega v}{\mu};\ W = \frac{2}{3\pi} + \frac{1+R}{\pi} \end{aligned} \tag{5.50}$$

The following analysis is absolutely similar to that in section 5.1. If $|u_0| > WA_{20}$, the only possibility is rotational motion of the pendulum, which corresponds to the break off of the resonance. But if $|u_0| < WA_{20}$, either rotation (also resulting in the resonance' break off) or oscillations are possible depending on the initial conditions. In other words the necessary condition for the existence of the stationary resonance is

$$|u_0| < WA_{20} \tag{5.51}$$

If this condition is fulfilled, the system (5.48) has two different stationary solutions with the same amplitude:

$$\begin{aligned} &A_{20} = \sqrt{\frac{4\varepsilon\omega^2 u_0}{3\pi\beta_e W}};\sigma_{20} = 0; \\ &\theta_{20}{}^{(1)} = -\arcsin\left(\frac{u_0}{WA_{20}}\right) \quad \text{or} \quad \theta_{20}{}^{(2)} = \pi + \arcsin\left(\frac{u_0}{WA_{20}}\right) \end{aligned} \tag{5.52}$$

The necessary condition for the existence of the resonance (3.3.34) can be rewritten now as follows:

$$0 < u_0 < \sqrt{\frac{4\varepsilon\omega^2 u_0 W}{3\pi\beta_e}} \tag{5.53}$$

It is obvious from the classical equation (5.50), that only the second solution can be stable (it corresponds to the stable equilibrium point of the equivalent pendulum).

Averaging with respect to semi-slow oscillations of the equivalent pendulum doesn't differ from the analysis in the section 5.1.4. Corresponding details can be found in [39].

5.3 Nonlinear Resonant Crusher with Inelastic Collisions

5.3.1 Problem Description. Equations of Motion

Consider the system shown in Fig. 5.10. It consists of a frame of mass M_1 in whose interior, which has a length $2l$, a striker of a mass m moves freely. An inertial exciter characterized by the inertia J and a static imbalance moment S is placed on the base of mass M_2 and set up in rotation by an induction motor. The analysis below is confined to linear approximation of the torque at the motor's shaft:

$$P(\dot{\psi}) = U - V\dot{\psi}, U > 0, V > 0 \tag{5.54}$$

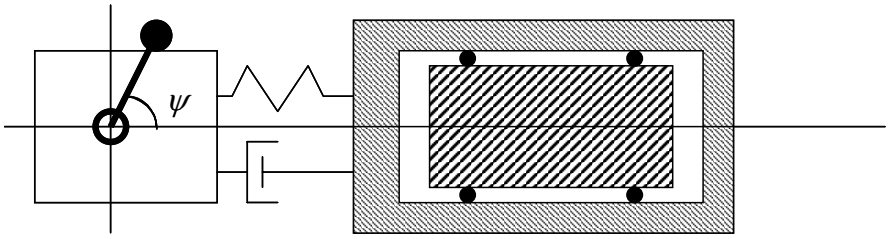

Fig. 5.10. Resonant crusher with inelastic collisions

This approximation is valid in the vicinity of the system's natural frequency

$$\lambda = \sqrt{c\left(\frac{1}{M_1 + m} + \frac{1}{M_2}\right)} \tag{5.55}$$

Here c is the stiffness of the spring connecting the base and the frame, the corresponding damping is b.

The system under consideration is a simple dynamical model of a resonant crusher. The base with the induction motor is separated from the frame in order to protect the bearings from the impact loads (*cf.* the previous section). The collisions

between the frame and the striker crush the stones of different sizes between them. The energy consumption for crushing is large. Thus we can consider the collisions as absolutely inelastic.

Let us introduce the following variables:

- x_1 is the coordinate of the centre of mass of the impact element composed of the frame and the striker;
- y is the relative coordinate of the striker in the clearance;
- x_2 is the coordinate of the base carrying the unbalanced exciter;
- ψ is the rotation angle of the exciter.

Then the equations of motion and the kinematical conditions for the inelastic collisions can be formulated as follows:

$$\begin{aligned}
\ddot{x}_1 &= \frac{c}{M_1+m}\left(x_2 - x_1 + \frac{m}{M_1+m}y\right) \\
&\quad + \frac{b}{M_1+m}\left(\dot{x}_2 - \dot{x}_1 + \frac{m}{M_1+m}\dot{y}\right) \\
\ddot{x}_2 &= -\frac{c}{M_2}\left(x_2 - x_1 + \frac{m}{M_1+m}y\right) \\
&\quad - \frac{b}{M_2}\left(\dot{x}_2 - \dot{x}_1 + \frac{m}{M_1+m}\dot{y}\right) - \frac{S}{J}(\cos\psi)^{\bullet\bullet} \qquad (5.56) \\
\ddot{\psi} &= \frac{U - V\dot{\psi}}{J} + \frac{S}{J}\ddot{x}_2 \sin\psi \\
\ddot{y} &= \begin{cases}
-\dfrac{c}{M_1}\left(x_2 - x_1 + \dfrac{m}{M_1+m}y\right) \\
\quad -\dfrac{b}{M_1}\left(\dot{x}_2 - \dot{x}_1 + \dfrac{m}{M_1+m}\dot{y}\right), & \text{for} \quad |y| < l \\
0, \text{ for} \quad |y| = l
\end{cases}
\end{aligned}$$

Equations (5.56) must be supplemented by separation conditions, which describe transitions from the joint to separate motion stages of the colliding partners:

$$c\left(x_1 - x_2 + \frac{m}{M_1 + m}y\right) + b\left(\dot{x}_1 - \dot{x}_2 + \frac{m}{M_1 + m}\dot{y}\right) = 0 \qquad (5.57)$$
$$\text{for } |y| = l, \quad \dot{y} = 0$$

Equations (5.56) and (5.57) describe motions of the system completely. It is sensible now to eliminate the degree of freedom corresponding to the motion of the center of mass of the whole system and to convert to undimensioned variables and parameters:

$$\mu = \frac{m}{M_1 + m}; \varepsilon = \frac{b}{2\lambda}\left(\frac{1}{M_1 + m} + \frac{1}{M_2}\right); u = \frac{U}{\lambda^2 J}; v = \frac{V}{\lambda J}$$
$$\delta = \frac{S}{lM_2}; W = \frac{l^2 M_2}{J}; k_1 = \frac{c}{M_1\lambda^2}; k_2 = \frac{c}{M_2\lambda^2} \qquad (5.58)$$
$$0 < k_1 < 1; \quad 0 < k_2 < 1$$
$$x = x_1 - x_2$$

Now the equations (5.56) and the separation conditions (5.57) can be rewritten as follows:

$$\ddot{x} = -\lambda^2\left(x - \mu y\right) - 2\lambda\varepsilon\left(\dot{x} - \mu\dot{y}\right) + \delta l\left(\cos\psi\right)^{\bullet\bullet}$$
$$\ddot{\psi} = \lambda^2 u - \lambda v\dot{\psi}$$
$$+ \frac{\delta W}{l}\left\{\lambda^2 k_2\left(x - \mu y\right) + 2\lambda\varepsilon k_2\left(\dot{x} - \mu\dot{y}\right) + \delta l\left(\cos\psi\right)^{\bullet\bullet}\right\} \qquad (5.59)$$
$$\ddot{y} = \begin{cases} \lambda^2 k_1\left(x - \mu y\right) + 2\lambda\varepsilon k_1\left(\dot{x} - \mu\dot{y}\right), & \text{for } |y| < l \\ 0, & \text{for } |y| = l \end{cases}$$

$$\lambda\left(x - \mu y\right) + 2\varepsilon\left(\dot{x} - \mu\dot{y}\right) = 0, \quad \text{for} \quad |y| = l, \quad \dot{y} = 0 \qquad (5.60)$$

Let us notice that the variable x_2 does not appear in the system (5.59), (5.60). It can de found from the second equation of the system (5.56) after the equations (5.59) and (5.60) are solved.

We suppose now $\varepsilon, \mu, \delta, u, v$ to be small parameters of the same magnitude order and omit the terms $o(\varepsilon)$:

$$\begin{aligned}
&\ddot{x} = -\lambda^2\left(x-\mu y\right) - 2\lambda\varepsilon\dot{x} - \delta l\ddot{\psi}\cos\psi \\
&\ddot{\psi} = \lambda^2 u - \lambda v\dot{\psi} + \frac{\delta W}{l}\left\{\lambda^2 k_2 x - \delta l\ddot{\psi}\cos\psi\right\} \\
&\ddot{y} = \begin{cases} \lambda^2 k_1\left(x-\mu y\right) + 2\lambda\varepsilon k_1\dot{x}, & \text{for } |y| < l \\ 0, & \text{for } |y| = l \end{cases} \\
&\lambda\left(x-\mu y\right) + 2\varepsilon\dot{x} = 0, \quad \text{for} \quad |y| = l, \quad \dot{y} = 0
\end{aligned} \tag{5.61}$$

System (5.61) shows that both variables x and ψ are continuous. The variable y is discontinuous and must be regularized in an appropriate way.

5.3.2 The Regularizing Transformation. The Main Resonance

The variety of motion regimes, which are possible in the considered system, is wide. It is determined by the sequence of the impacts. Below we shall restrict the analysis to an elementary regime characterized by the presence of contact zones (finite time intervals of joint motion of the striker and frame) with two collisions per frame's oscillation period. The following discontinuous transformation can be applied in order to regularize the variable y (*cf.* the discontinuous transformation for systems with inelastic collisions discussed in section 4.4):

$$\begin{aligned}
&x = A\sin\chi; \quad \dot{x} = A\lambda\cos\chi; \quad \dot{\psi} = \lambda\Omega \\
&y = lL\left(\varphi,\alpha\right) - Ak_1\sin\chi M_1\left(\varphi,\alpha\right) \\
&\quad + \left\{-l + B\left(\varphi - \pi\left[\frac{\varphi}{\pi}\right]\right)\right\} M_1\left(2\varphi,\alpha\right)\operatorname{sgn}\sin\varphi \\
&\dot{y} = \lambda\left\{B\operatorname{sgn}\sin\varphi - Ak_1\cos\chi\right\} M_1\left(2\varphi,\alpha\right) \\
&L\left(\psi,\alpha\right) = \left(1 - M_1\left(2\varphi,\alpha\right)\right)\operatorname{sgn}\sin\varphi
\end{aligned} \tag{5.62}$$

Here, as usual, $\left[z\right]$ means the integer part of z, function M_1 is defined as follows:

$$M_1\left(\varphi,\alpha\right) = \begin{cases} 1, & 0 < \varphi < \alpha \\ 0, & \alpha < \varphi < 2\pi \end{cases}; \quad M_1\left(\varphi + 2\pi,\alpha\right) = M_1\left(\varphi,\alpha\right) \tag{5.63}$$

Variable α is determined by the transcendental condition corresponding to the time point of a collision:

$$B\alpha - Ak_1 \sin\chi = 2l, \quad \text{when } \varphi = \alpha \tag{5.64}$$

New variables are governed by the following equations:

$$\begin{aligned}
&\dot{A} = \lambda\mu y \cos\chi - 2\lambda\varepsilon S\cos^2\chi - \delta\lambda \quad l\Omega^2 \cos\chi\cos\psi \\
&\dot{\chi} = \lambda + \lambda\varepsilon \sin 2\chi - \frac{\lambda}{A}\left(\mu y - \delta \quad l\Omega^2 \cos\psi\right)\sin\chi \\
&\dot{\psi} = \lambda\Omega \\
&\dot{\Omega} = \lambda u - \lambda v\Omega + \frac{\delta W}{l}\lambda k_2 A \sin\chi \sin\psi \\
&\dot{B}M_2(\varphi,\alpha) = -\lambda\delta k_1 l\Omega^2 \cos\psi M_2{}^2(\varphi,\alpha) \\
&\dot{\varphi}M_2(\varphi,\alpha) = \lambda M_2(\varphi,\alpha) \\
&\qquad + \frac{\lambda k_1 \delta \quad l}{B}\Omega^2\left(\varphi - \pi\left[\varphi/\pi\right]\right)\cos\psi M_2{}^2(\varphi,\alpha) \\
&M_2(\varphi,\alpha) = M_1(2\varphi,\alpha)\operatorname{sgn}\sin\varphi
\end{aligned} \tag{5.65}$$

It is assumed that y and $\dot{y}$ are expressed here in terms of the new phase variables. We note that the above transformation of variables guarantees identical satisfaction of the equations during the contact motion stages, $|y| = l$. In the equations (5.65) this circumstance is manifested in the fact that the right- and the left-hand sides of the last two equations vanish identically during these stages. The variables B and φ as functions of time are therefore solutions of a sequence of differential equations for which the new initial conditions (5.60) are adopted at times determined by the separation conditions.

The main principal resonance in the system (5.65) corresponds to the surface $\Omega = 1$. Two semi-slow variables and one slow variable can be introduced in the $\sqrt{\varepsilon}$ -vicinity of the resonance:

$$\theta = \varphi - \chi; \quad \sigma = \frac{\Omega - 1}{\sqrt{\mu}}; \quad \eta = \psi - \chi; \quad \tau = \lambda t \tag{5.66}$$

Substituting (5.66) into (5.65) one obtains equations in the special form which was discussed in Chapter 4. It contains one slow variable A, two semi slow variables σ and η, two slave variables θ and B, and one fast rotating phase χ.

$$
\begin{aligned}
A' &= \mu\cos\chi\left\{y - 2\varepsilon_1 A\cos\chi - 2\delta_1 l\cos\left(\chi+\eta\right)\right\}\\
\sigma' &= \sqrt{\mu}\left\{u_1 - v_1 + 2\delta_1 W l^{-1} k_2 A\sin\chi\sin\left(\chi+\eta\right)\right\} - \mu v_1\sigma\\
\eta' &= \sqrt{\mu}\sigma + \mu A^{-1}\sin\chi\left\{y - 2\varepsilon_1 A\cos\chi - 2\delta_1 l\cos\left(\chi+\eta\right)\right\}\\
\theta' M_2\left(\chi+\theta,\alpha\right) &= \mu M_2\left(\chi+\theta,\alpha\right)\left\{yA^{-1}\sin\chi - \varepsilon_1\sin 2\chi\right.\\
&\quad - 2\delta_1 l A^{-1}\sin\chi\cos\left(\chi+\eta\right)\\
&\quad \left. + 2k_1\delta_1 l B^{-1}\left(\chi+\theta-\pi\left[\frac{\chi+\theta}{\pi}\right]\right)\cos\left(\chi+\eta\right)M_2\left(\chi+\theta,\alpha\right)\right\}\\
B' M_2\left(\chi+\theta,\alpha\right) &= -2\mu\delta_1 l k_1\cos\left(\chi+\eta\right)M_2{}^2\left(\chi+\theta,\alpha\right)\\
\chi' &= 1 - \mu A^{-1}\sin\chi\left\{y - 2\varepsilon_1 A\cos\chi - 2\delta_1 l\cos\left(\chi+\eta\right)\right\}
\end{aligned}
\tag{5.67}
$$

The separation conditions take the form

$$
\chi\left(t_n\right) = n\pi,\quad n = 0,1,2,\ldots,\quad \theta\left(t_n\right) = 0,\quad B\left(t_n\right) = k_1 A\left(t_n\right) \tag{5.68}
$$

The slow variable α is determined by the equation

$$
B\alpha - Ak_1\sin\left(\alpha-\theta\right) = 2l \tag{5.69}
$$

In order to use only one small parameter, we have introduced here the following notation

$$
\varepsilon_1 = \varepsilon/\mu;\quad \delta_1 = \delta/2\mu;\quad u_1 = u/\mu;\quad v_1 = v/\mu \tag{5.70}
$$

These equations can be averaged with respect to the fast rotating phase χ.

5.3.3 Averaging with Respect to the Fast Rotating Phase. Stationary Regimes

The result of the averaging (the second order approximation) is as follows:

$$
\begin{aligned}
A_2' &= -\mu\left(\varepsilon_1 A_2 - \delta_1 l\cos\eta_2 + lF\left(A_2/l\right)\right)\\
\eta_2' &= \sqrt{\mu}\sigma_2 + \mu\left(\delta_1 l A_2^{-1}\sin\eta_2 + G\left(A_2/l\right)\right)\\
\sigma_2' &= \sqrt{\mu}\left(u_1 - v_1 + \delta_1 l^{-1} W k_2\cos\eta_2\right) - \mu v_1\sigma_2\\
B_2 &= k_1 A_2;\ \theta_2 = 0
\end{aligned}
\tag{5.71}
$$

Here we have introduced the following notation (subscript 2 is attached to the corresponding averaged variables):

$$F\left(A_2/l\right)=\frac{A_2k_1}{2\pi l}\left(1-\cos^2\alpha_2\right)$$
$$G\left(A_2/l\right)=\frac{k_1}{2\pi}\left(\alpha_2-\left(2-\cos\alpha_2\right)\sin\alpha_2\right) \tag{5.72}$$
$$\alpha_2-\sin\alpha_2=\frac{2l}{A_2k_1}$$

In order to simplify the analysis of (5.71) the following notation can be introduced:

$$a_2=\frac{A_2}{l};\ \ \xi_2=\eta_2-\frac{\pi}{2};\ \ W_2=k_2W\delta_1;\ u_0=u_1-v_1 \tag{5.73}$$

Neglecting the small terms $O\left(\mu\right)$ in the equations (5.71) we obtain the well known equation describing a pendulum with the applied external torque:

$$\left.\begin{aligned}a_{20}'&=0\\ \xi_{20}'&=\sqrt{\mu}\sigma_{20}\\ \sigma_{20}'&=\sqrt{\mu}\left(u_0-W_2a_{20}\sin\xi_{20}\right)\end{aligned}\right\}\Leftrightarrow \xi_{20}''-\mu W_2a_{20}\sin\xi_{20}=\mu u_0 \tag{5.74}$$

The following analysis is absolutely similar to that in sections 5.1 and 5.2. If $\left|u_0\right|>W_2a_{20}$, the rotation of the pendulum is the only possibility, which corresponds to the break off of the resonance. But if $\left|u_0\right|<W_2a_{20}$, either rotation (also resulting in the resonance' break off) or oscillations are possible depending on the initial conditions. In other words the necessary condition for the existence of the stationary resonance is

$$\left|u_0\right|<W_2a_{20} \tag{5.75}$$

If this condition is fulfilled, the system (5.71) has two different stationary solutions with the amplitude determined by the equation:

$$\varepsilon_1a_2^2+a_2F\left(a_2\right)=\frac{\delta_1u_0}{W_2} \tag{5.76}$$

These solutions differ through their phases:

$$\xi_{20}^{(1)} = -\arcsin\left(\frac{u_0}{W_2 a_{20}}\right) \quad \text{or} \quad \xi_{20}^{(2)} = \pi + \arcsin\left(\frac{u_0}{W_2 a_{20}}\right) \tag{5.77}$$

It is obvious from the classical equation (5.74), that only the second solution can be stable (it corresponds to the stable equilibrium point of the equivalent pendulum).

Averaging with respect to semi slow oscillations of the equivalent pendulum doesn't differ significantly from the analysis in the section 5.1.4. The corresponding details can be found in [87].

It should be added that all the above analysis must be supplemented by an existence condition for the considered impact regime:

$$0 < \alpha_2 < \pi \text{ or } a_2(t) > \frac{2\pi}{k_1} \tag{5.78}$$

If this condition is not fulfilled another regime with less frequent collisions or possibly "continuous juggling" will appear. In many practical cases, regimes with less frequent collisions are just as undesirable, as the resonance breakdown with sharply decreasing vibration amplitude is. In order to avoid this situation the following relationships should be taken into account. If $a_{20} < \frac{2}{\pi k_1}$ the stationary motion of the considered type is impossible. The attraction region of the stationary resonance is maximal, i.e. it corresponds to the internal homoclinical loop in Fig. 5.4, if the following condition holds:

$$\frac{2}{\pi k_1} < \frac{u_0}{W_2} \tag{5.79}$$

Otherwise our analysis is not valid in a part of this loop and we cannot guarantee that the integral curves from this part will reach the stationary point.

5.4. Conclusions

Significantly nonlinear resonance is an extremely wide, important and complex research field both from practical and from theoretical point of view. It occurs in all systems where at least one frequency (either the "natural frequency" or the frequency of excitation) is not constant, but depends significantly on the unknown solution. The main peculiarity of the nonlinear resonance is its wide attraction area. It spreads in a large ($O\left(\sqrt{\varepsilon}\right)$) area around the resonant surface. Dynamics of systems in the vicinity of the nonlinear resonance is also complex. It is necessary to distinguish between fast motions corresponding to excitation and "visible"

oscillations of the system, semi slow oscillations of the phase differences in the resonance domain and slow evolution of the general characteristics of the system, like its energy or the amplitude of the fast oscillations.

Standard averaging with $\sqrt{\varepsilon}$ considered as the small parameter is sufficient for investigating the stationary resonant regimes and their stability, but it does not allow finding the attraction area of the stable resonances in the phase space, because it is valid within an insufficient time interval. However, the system averaged with respect to both fast and semi slow oscillations in the resonance domain is valid within a sufficiently long time interval and is therefore suitable for analyzing the transient motions.

The described complex dynamics is especially important if the technical objective is not to avoid the resonance, but to create and to stabilize intensive vibrations. Then the semi slow phase dynamics is the key point for the development of any efficient control.

The described approach can be easily combined with the averaging procedures established for discontinuous systems. This combination is especially important because resonant machines are often user for crushing, grinding, screening and transporting of bulk materials in ore processing and chemical industries.

6. High Frequency Excitation: Basic Ideas and Elementary Effects

Systems with high frequency (HF) motions appear everywhere in physics, engineering and common life. The conventional approach to systems with HF motions is to ignore them completely. Examples for this approach can be found everywhere. If we are talking about motions of a rigid body we ignore the HF elastic waves which are an unavoidable companion for any acceleration or deceleration. Investigating elastic waves we ignore the HF thermal oscillations of the molecules this body is built of. Analyzing the molecules' motions we usually ignore the HF electron waves replacing them by some kind of imagined linear or nonlinear springs connecting particular atoms. Talking with a friend we usually ignore the nonlinear electro-magnetic waves in his brain even though without these waves no discussion would take place.

These examples show on the one hand that the ignorance towards HF motions is one of the fundamentals of our science and culture. Usually we do not even notice what we ignore in a particular situation.

On the other hand they permit an idea what is usually meant if one is talking about HF motions. Under HF motions we understand oscillations if their frequency significantly exceeds the frequencies of the processes we are interested in. It is important that the general term "motions" is replaced here through the term "oscillations" being a particular kind of a motion. The reason becomes obvious if we look back at the systems with collisions discussed in the previous chapters. Any collision itself is definitely much faster than the oscillations we were interested in, but on the other hand it could never be neglected in investigated problems.

There are many reasons why we ignore the HF motions. The first one is that our cognition is originated in human senses. In the common life we are used to ignore things which we cannot see, hear, smell or touch. The second and much more important reason is that without ignorance, without simplifying modeling the cognition and science would hardly be possible at all. In many cases it would be totally senseless to try to get some useful information about system's behavior by following each thrill of each component when we are really interested in some generalized or averaged characteristics.

The mathematical basis for our ignorance is given through the strong filtering properties distinctive for many systems and sensors. However it is well known that this kind of ignorance is not always useful and correct. If a nonlinear system is excited in the HF domain and the excitation is sufficiently strong, the system's properties with respect to slow motions can be changed significantly. The obvious

examples are phase transitions due to increased or decreased temperature like melting ice or demagnetized iron. Numerous examples of interaction between HF excitation and system's response at a low frequency can be found in pure mechanical systems too. The stabilizing of the upside-down position of a simple pendulum due to invisible vibrations with small amplitude and a frequency much higher than the natural frequency of the corresponding linearized system is one of the well known examples. Another one is the smoothing of the dry friction in presence of sufficiently strong tangential vibrations. The result of this smoothing is the ability of HF vibrations to quench the self excited oscillations in systems with negative friction gradient. These two elementary examples are considered in sections 6.2 and 6.3. Before that the classification of dynamic systems with HF excitation is given in section 6.1. The problem of shifted resonances of a pendulum is discussed in paragraph 6.4. The exceptional role of the terms depending on the velocities is discussed in paragraph 6.5. The effect of HF vibrations (dither) in strongly damped control systems is investigated finally in section 6.6 where some advanced ideas like averaging observer are also discussed.

Examples illustrating general properties of mechanical systems under HF excitation can be found in almost any book discussing nonlinear oscillations [23, 24, 79, 80]. The main reference in this area is surely the monograph by Blekhman [20]. An excellent classification of the contemporary used effects was done by Thomsen [119]. Some additional examples can be found in [81, 120, 121 – 125].

Mathematical basics for the averaging of systems with HF excitation are discussed in Chapter 7. Several advanced examples concentrated on the effect of the oscillating terms containing the generalized velocities are also investigated there. This chapter is strongly recommended not only to mathematicians but also to those who are interested in why the analysis of the HF vibrations can influence our world-outlook. We live at the threshold of the new scientific revolution. The deep understanding of interactions between different scales in time and space domains will be one of the main scientific challenges in the coming decades.

6.1 Classification of Systems with HF Excitation. Weakly Excited Systems

6.1.1 Classification of Systems with HF Excitation

In this chapter we are going to consider systems of "*mechanical type*", i.e. systems described by second order differential equations as follows:

$$\ddot{x} = \omega^{\alpha}\Phi\left(x, \dot{x}, t, \omega t\right) \tag{6.1}$$

Here x is an n-dimensional vector of the generalized coordinates, $\dot{x}$ is the corresponding vector of the generalized velocities, $\omega >> 1$ is a large parameter.

We assume Φ to be an n-dimensional vector of forces which depends 2π-periodically on the fast time $\tau = \omega t$. The presence of the terms depending on the fast time expresses the presence of the HF excitation.

Depending on the magnitude of the integer parameter α one can distinguish between systems with weak, strong and very strong excitation:

- The HF excitation is weak for $\alpha = 0$
- It is strong for $\alpha = 1$
- It is very strong for $\alpha = 2$

6.1.2 Systems with Weak HF Excitation

In this section we consider the trivial case of the weak excitation. It illustrates clearly, why the HF excitation and the corresponding motions are usually neglected.

The equations (6.1) can be easily transformed to the standard form for averaging. Converting to the fast time τ as a new independent variable and rewriting (6.1) as a system of $2n$ first order differential equations we obtain:

$$\begin{aligned} x' &= \varepsilon y \\ y' &= \varepsilon \Phi\left(x, y, t, \tau\right) \\ t' &= \varepsilon \\ \varepsilon &= \omega^{-1} << 1 \end{aligned} \tag{6.2}$$

These equations can be averaged directly if the function Φ is sufficiently smooth. The equations of the first order approximation can be written as follows:

$$\begin{aligned} x_1' &= \varepsilon y_1 \\ y_1' &= \varepsilon \left\langle \Phi\left(x_1, y_1, t, \tau\right) \right\rangle \\ t' &= \varepsilon \end{aligned} \tag{6.3}$$

If we return back to t as the independent variable, we obtain the final averaged equation:

$$\ddot{x}_1 = \left\langle \Phi\left(x_1, \dot{x}_1, t, \tau\right) \right\rangle \tag{6.4}$$

Here and further $\left\langle \Phi \right\rangle$ means the average of Φ with respect to the fast time τ. The solution to (6.4) is asymptotically close to the solution of the original system (6.1) in the time interval $\tau = O\left(\omega\right)$ or $t = O\left(1\right)$.

The obtained result is very simple. The solution to the system with the weak HF excitation is a superposition of small fast oscillations and slow evolution of the system. In many cases these components are almost independent.

6.1.3 The Weakly Excited Pendulum

In order to illustrate this statement let us investigate a pendulum with the vertically oscillating suspension point (see Fig. 6.1). Its motion is governed by the following equation:

$$ml^2 \frac{d^2\theta}{d\tilde{t}^2} + 2b\frac{d\theta}{d\tilde{t}} + mgl\sin\theta = mlu\Omega^2 \sin\Omega\tilde{t}\sin\theta \tag{6.5}$$

This equation can be transformed to the dimensionless form if we introduce the following parameters and variables:

$$\omega_0 = \sqrt{\frac{g}{l}};\ \omega = \frac{\Omega}{\omega_0};\ t = \omega_0\tilde{t};\ \beta = \frac{b}{ml^2\omega_0};\ a = \frac{u}{l};\ \dot{\theta} = \frac{d\theta}{dt} \tag{6.6}$$

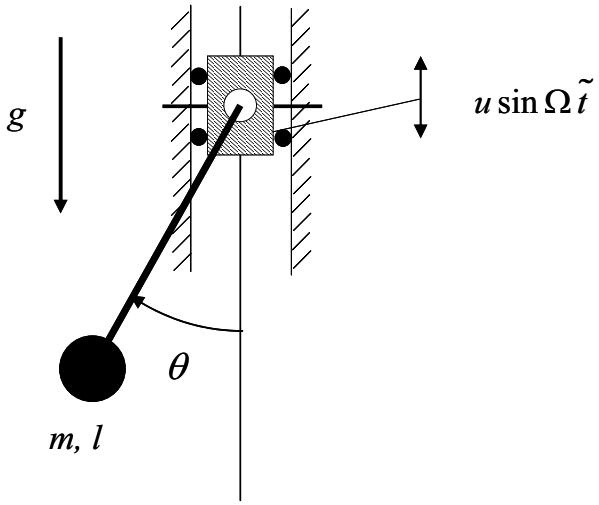

Fig. 6.1. Pendulum with a vertically vibrating suspension point

This transformation leads to the following equation:

$$\ddot{\theta} + 2\beta\dot{\theta} + \sin\theta = a\omega^2 \sin\omega t\sin\theta \tag{6.7}$$

This equation contains three dimensionless parameters: a, β and ω. In this and the next chapter we shall always presume ω to be a large parameter. In this

section we presume additionally that damping is sufficiently small $\beta = O(1)$ and the prescribed support oscillations' amplitude is so small that the product $a\omega^2 = q = O(1)$, i.e. $a = O(\omega^{-2})$. All the assumptions of our analysis are valid as long as these requirements are fulfilled. The equation (6.7) can be averaged directly. The result is as trivial as it was expected. It describes the motions of the same pendulum without any external excitation:

$$\ddot{\theta}_1 + 2\beta\dot{\theta}_1 + \sin\theta_1 = 0 \tag{6.8}$$

In other words, the weak HF excitation and the corresponding motions can be ignored at least with the accuracy corresponding to the first order approximation. The mistake has the order $O(\omega^{-1})$, i.e. it decreases with the increasing excitation's frequency if the excitation's intensity q remains constant.

This result is illustrated in Fig. 6.2 and Fig. 6.3, where the simulation results are displayed for both an excited and unexcited pendulum for the following parameters values: $\beta = 0.1;\ \omega = 10;\ a = 0.01\ (q = 1)$. The simulations were performed for the same initial conditions: $\theta(0) = 3;\ \theta'(0) = 0$. The results are shown for both the swing angle of the pendulum and its velocity.

It is interesting to notice that there is no visible difference between the solutions to these systems at the coordinate level. A small difference remains visible if we compare the velocities. This small difference being seemingly insignificant here becomes more and more important if we increase the excitation's intensity. The principal difference between coordinates and velocities in the mechanical systems under HF excitation will be explicitly discussed in section 6.4. Now let us move on and discuss the same pendulum under the strong HF excitation.

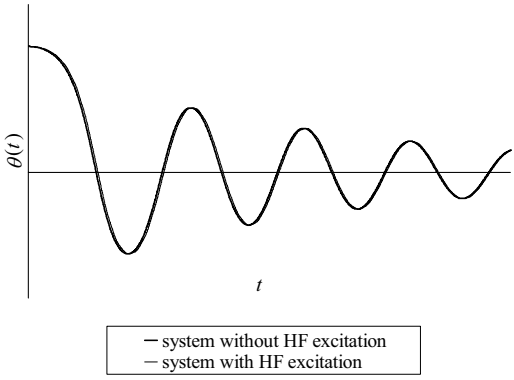

Fig. 6.2. Comparison between the swing angle of a pendulum without HF excitation and the same pendulum in the presence of a weak HF excitation

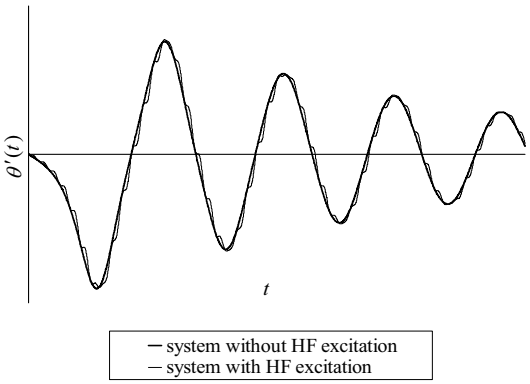

Fig. 6.3. Comparison between the velocities

6.2 A Strongly Excited Pendulum with the Oscillating Suspension Point. Stiffening, Softening and Biasing

6.2.1 A Pendulum with the Vertically Vibrating Suspension Point: Equations Governing the Slow Motions

The same pendulum as in the previous section is investigated here but we change our assumptions about the intensity of the external excitation. In other words, the system is still governed by the equations (6.7) but now we suppose:

$$a = O\left(\omega^{-1}\right);\ a\omega = A = O\left(\omega\right) \tag{6.9}$$

It means in terms of the previous section the pendulum is excited strongly. Let us convert to the fast time as a new independent variable:

$$\begin{aligned} &\theta'' + 2\beta\varepsilon\theta' + \varepsilon^2 \sin\theta = a\sin\tau\sin\theta \\ &\tau = \omega t;\ \ \varepsilon = \omega^{-1} << 1;\ \ \theta' = \frac{d\theta}{d\tau} \end{aligned} \tag{6.10}$$

Now we can convert from the second order differential equation to a system of two first order equations as follows:

$$\begin{aligned} &\theta' = \varepsilon\rho \\ &\rho' = -2\beta\varepsilon\rho - \varepsilon\sin\theta + A\sin\tau\sin\theta \end{aligned} \tag{6.11}$$

The unperturbed system corresponding to (6.11) is:

$$\begin{aligned} \theta_0' &= 0 \\ \rho_0' &= A\sin\tau\sin\theta_0 \end{aligned} \tag{6.12}$$

Its general solution is easy to obtain:

$$\begin{aligned} \theta_0 &= const \\ \rho_0 &= -A\cos\tau\sin\theta_0 + \sigma \end{aligned} \tag{6.13}$$

Using this solution to the unperturbed system we can apply the following transformation in order to get a system in the standard form for averaging:

$$\begin{aligned} &\rho = -A\cos\tau\sin\theta + \sigma \;\Rightarrow \\ &\theta' = \varepsilon\sigma - \varepsilon A\cos\tau\sin\theta \\ &\sigma' = -\varepsilon\sin\theta + \varepsilon\left(A\cos\tau\cos\theta - 2\beta\right)\left(\sigma - A\cos\tau\sin\theta\right) \end{aligned} \tag{6.14}$$

These equations can be averaged directly:

$$\theta_1'' + 2\beta\varepsilon\theta_1' + \varepsilon^2\sin\theta_1 + \frac{1}{2}\varepsilon^2 A^2\sin\theta_1\cos\theta_1 = 0 \tag{6.15}$$

Returning back to the slow time as the independent variable we obtain the equation governing the slow motions of the HF excited pendulum (first order approximation):

$$\ddot{\theta}_1 + 2\beta\dot{\theta}_1 + \sin\theta_1 + \frac{1}{4}A^2\sin 2\theta_1 = 0 \tag{6.16}$$

6.2.2 Discussion of the Results for the Vertically Excited Pendulum

The equation (6.16) is very interesting from many points of view. First of all it is an autonomous system. It does not contain the explicit time. On the other hand it differs from the equation for the unexcited or weakly excited pendulum (6.8) through the last term $\frac{1}{4}A^2\sin 2\theta_1$. This term represents the effect of the HF excitation on the slow motions of our system and is called "*the vibrational force*" (compare [20] and Chapter 5). If we neglect the damping, (6.16) would be a conservative system. Its main properties can be described by the corresponding potential function:

$$\ddot{\theta}_1 + 2\beta\dot{\theta}_1 + \frac{\partial \Pi}{\partial \theta_1} = 0$$
$$\Pi = 1 - \cos\theta_1 + \frac{1}{8}A^2\left(1 - \cos 2\theta_1\right) \tag{6.17}$$

The free constant in the potential function is chosen here in order to assure $\Pi(0) = 0$. This function for different values of the excitation's intensity A is shown in Fig. 6.4.

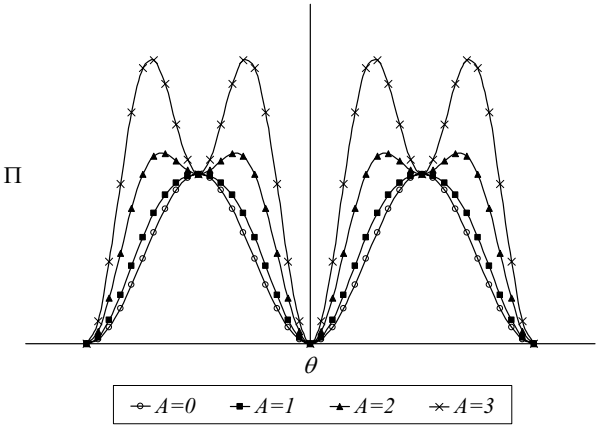

Fig. 6.4. Potential function for the slow motions of the vertically HF excited pendulum

The pendulum without HF excitation (i.e. for $A = 0$) has two equilibrium points: the down-pointing equilibrium $\theta = 0$ and the up-pointing one $\theta = \pi$. The first one corresponds to the minimum of the potential energy and is stable; the second one corresponds to the maximum and is consequently unstable. The effective stiffness of the pendulum in the vicinity of its stable equilibrium position is equal to one (according to our undimensioned notation). This effective stiffness increases in presence of the HF excitation:

$$C^0_{eff} = 1 + \tfrac{1}{2}A^2 = 1 + \tfrac{1}{2}a^2\omega^2 \tag{6.18}$$

The same statement is valid for the effective frequency of small slow oscillations in the vicinity of this point:

$$\omega^0_{eff} = \sqrt{1 + \tfrac{1}{2}a^2\omega^2} \tag{6.19}$$

This effect is natural to call "*stiffening*" [119].

A similar interpretation can be given for the originally unstable up-pointing equilibrium position. The corresponding effective stiffness and the effective frequency also increase in presence of the HF excitation:

$$C_{eff}^{\pi} = \tfrac{1}{2}a^2\omega^2 - 1; \;\; \omega_{eff}^{\pi} = \sqrt{\tfrac{1}{2}a^2\omega^2 - 1} \tag{6.20}$$

The up-pointing equilibrium becomes stable as soon as this stiffness becomes positive, i.e. for $a\omega > \sqrt{2}$. This threshold value guarantees also that the effective frequency becomes real. This change is connected with the appearance of two additional equilibriums corresponding to the maxima of the potential energy. These equilibriums are obviously unstable.

6.2.3 The Pendulum With the Horizontally Vibrating Suspension Point: Equations of Slow Motions and System's Behavior

The situation changes slightly if we consider the same pendulum which the suspension point oscillates in the horizontal direction (see Fig. 6.5). Its equations of motion are similar to the case of the vertical excitation. In the undimensioned form these equations can be written as follows:

$$\begin{gathered} \theta'' + 2\beta\varepsilon\theta' + \varepsilon^2 \sin\theta = b\sin\tau\cos\theta \\ \tau = \omega_1 t; \;\; \omega_1 = \frac{\Omega_1}{\omega_0}; \;\; b = \frac{v}{l} \end{gathered} \tag{6.21}$$

The whole analysis is also similar to the previous case. The equation of the first order approximation which governes the slow motions of the pendulum is

$$\ddot{\theta}_1 + 2\beta\dot{\theta}_1 + \sin\theta_1 - \frac{1}{4}B^2 \sin 2\theta_1 = 0; \;\; B = b\omega_1 \tag{6.22}$$

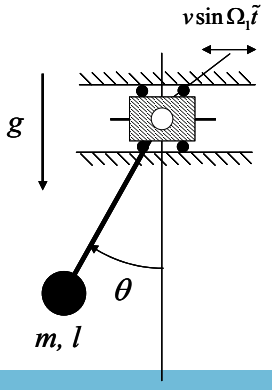

Fig. 6.5. Pendulum with the horizontally vibrating suspension point

The only difference between this case and the system with the vertical HF excitation is the sign of the vibrational force. Now it is always negative. It reveals itself in the corresponding potential energy of the averaged system, which is shown in Fig. 6.6.

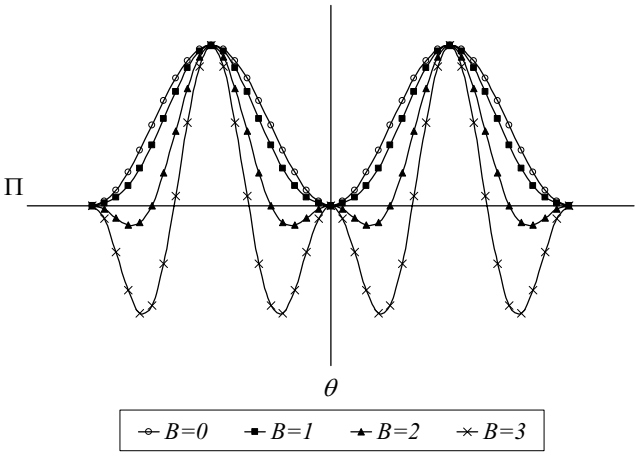

Fig. 6.6. Potential function for the slow motions of the horizontally HF excited pendulum

The effect of the HF vibrations is in this case different. The effective stiffness corresponding to the down-pointing equilibrium decreases with increasing intensity of the excitation:

$$C_{eff}^{0} = 1 - \tfrac{1}{2} B^2 = 1 - \tfrac{1}{2} b^2 \omega_1^{\,2} \tag{6.23}$$

This equilibrium becomes unstable, if this intensity is sufficiently strong ($b\omega_1 > \sqrt{2}$). It is quite natural to call this effect "*softening*" correlating this name with the stiffening in the previous case. The up-pointing equilibrium remains unstable for any excitation's intensity. But the lost stability of the down-pointing equilibrium results in two new symmetrically situated stable equilibriums appearing at the same threshold:

$$\theta_* = \pm \arccos\left(\frac{2}{b^2 \omega_1^{\,2}} \right) \tag{6.24}$$

Figure 6.7 shows the result of the numeric simulations for the pendulum. It is not excited initially. After some transient process the pendulum stabilizes itself in the down-pointing equilibrium. At the time point of 25 seconds the HF excitation is switched on. A new transient process starts. At its end the pendulum stabilizes itself in the new equilibrium and remains tilted. This effect is usually called "*bias-*

ing" (*cf.* [119]). The thick line for the pendulum's position is a result of its HF vibrations.

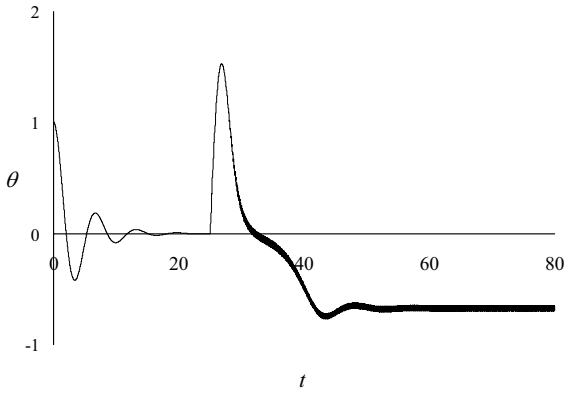

Fig. 6.7. Biasing of the pendulum due to horizontal HF excitation

This effect can be also considered as a control mechanism. Applying an appropriate HF excitation one can bias the pendulum from its original down-pointing position. The disadvantage of the control mechanism discussed above is the indefiniteness of the final stable position. According to (6.24) there are two stable equilibriums. Which one will be the final system's position depends on the initial conditions. This disadvantage can be easily avoided if we apply the two excitation types (vertical and horizontal) simultaneously.

6.2.4 The Pendulum Excited both Vertically and Horizontally

Consider a pendulum which suspension point is excited both in vertical and horizontal directions. The excitation's amplitudes are different ($a_{\parallel}$ for the vertical excitation and $a_{\perp}$ for the horizontal one) but the excitation's frequency is the same. We introduce in addition the phase difference between the excitations γ. Then the equation of motion for the excited pendulum in the undimensioned form can be written as follows:

$$\ddot{\theta} + 2\beta\dot{\theta} + \sin\theta = a_{\parallel}\omega^2 \sin\omega t \sin\theta + a_{\perp}\omega^2 \sin\left(\omega t + \gamma\right)\cos\theta \qquad (6.25)$$

The chosen excitation form means that the suspension point moves along an ellipse. The orientation of the ellipse and the direction of motion can be chosen arbitrarily (see Fig. 6.8).

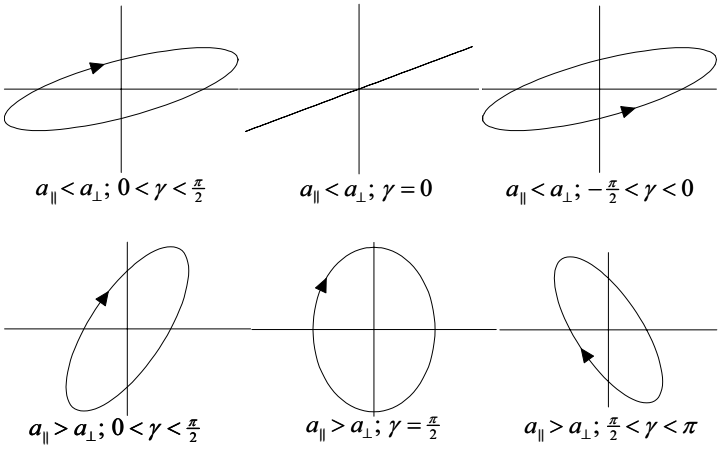

Fig. 6.8. Some possibilities to choose the motion of the suspension point

The same approach can be used in order to analyze the slow motions of the elliptically excited pendulum. Introducing the fast time as the new independent variable and converting to a system of two first order differential equations we obtain:

$$\begin{aligned} &\theta' = \varepsilon\rho \\ &\rho' = 2\beta\varepsilon\rho - \varepsilon\sin\theta + a_{\|}\omega\sin\tau\sin\theta + a_{\perp}\omega\sin\left(\tau+\gamma\right)\cos\theta \end{aligned} \tag{6.26}$$

The large terms in (6.26) can be eliminated if we apply the transformation based on the solution to the unperturbed problem:

$$\rho = \sigma - a_{\|}\omega\cos\tau\sin\theta - a_{\perp}\omega\cos\left(\tau+\gamma\right)\cos\theta \tag{6.27}$$

The equations for θ and σ are in the standard form and can be averaged directly. Returning back to the slow time one obtains the following equation governing the slow motions of the elliptically HF excited pendulum:

$$\begin{aligned} &\ddot{\theta} + 2\beta\dot{\theta} + \sin\theta = \\ &\qquad -\frac{1}{4}\left(a_{\|}^{\,2} - a_{\perp}^{\,2}\right)\omega^2\sin 2\theta - \frac{1}{2}a_{\|}a_{\perp}\omega^2\cos\gamma\cos 2\theta \end{aligned} \tag{6.28}$$

The corresponding potential function is:

$$\begin{aligned} \Pi = 1 - \cos\theta + \frac{1}{8}\left(a_{\|}^{\,2} - a_{\perp}^{\,2}\right)\omega^2\left(1-\cos 2\theta\right) \\ + \frac{1}{4}a_{\|}a_{\perp}\omega^2\cos\gamma\sin 2\theta \end{aligned} \tag{6.29}$$

Figure 6.9 shows the potential function for the particular case $a_{\|} = a_{\perp} = \dfrac{2}{\omega}$ for different values of the phase difference γ.

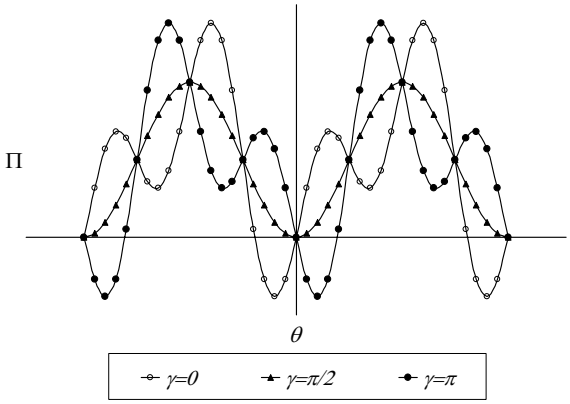

Fig. 6.9. Potential function for the slow motions of the elliptically HF excited pendulum

It is clear that it is possible not only to make the down pointing equilibrium unstable applying the directed HF excitation, but to bias the equilibrium to some different position. The down-pointing position in Fig. 6.9 for $\gamma = 0$ and $\gamma = \pi$ is not equilibrium at all. The pendulum can be tilted in one or another direction depending on the phase difference γ. It becomes also possible to create new stable equilibriums in the upper semi-circle, which differ from the up-pointing position. In the previously discussed cases it was impossible.

If we are interested in the bias of the pendulum from its down-pointing equilibrium, we can linearize the equation (6.28) in the vicinity of $\theta = 0$ and obtain the approximate expression for the new equilibrium in the presence of the HF excitation:

$$\theta_{st} = -\frac{a_{\|} a_{\perp} \omega^2 \cos\gamma}{2 + \left(a_{\|}^{\,2} - a_{\perp}^{\,2}\right)\omega^2} \tag{6.30}$$

The same result can be also obtained directly from the linearized equation (6.25). The result has a very characteristic structure. The stationary tilt is proportional to the product of the excitation's amplitudes. We will find a similar structure in a much more complex problem of a controlled pendulum which is discussed at the end of this chapter.

The effective stiffness in the new equilibrium is

$$c_{eff} = 1 + \tfrac{1}{2}\left(a_{\|}^{\,2} - a_{\perp}^{\,2}\right)\omega^2 \tag{6.31}$$

6.3 Shifted Resonances of the Pendulum. The Overlapped Slow Excitation and the Slowly Modulated HF Excitation

6.3.1 Two Types of Bi-harmonic Excitation

The described phenomenon of the stiffening and softening of a nonlinear system is deeply connected with the so called "*shifted resonances*". The basic idea of the shifted resonances is quite simple. Consider a system excited both at a slow and a high frequency. The influence of these two excitations cannot be analyzed separately. The strong HF excitation dominates. Stiffening or softening is a result. If the system with the effectively changed stiffness is excited slowly then one can expect it would behave resonantly at frequencies corresponding not to its original natural modes but at the frequencies corresponding to the new ones. It is really the case. This statement becomes clear if we investigate the pendulum excited both vertically and horizontally (see Fig. 6.10). We suppose the external excitations to contain both slow and fast components. There are two different types of slow components which are interesting for our analysis. The first one is an excitation which is added to the fast excitation, i.e. the fast excitation is overlapped by the slow component. This type of excitation is displayed in Fig. 6.11(a). The second one is the slowly modulated fast excitation. The corresponding example is shown in Fig. 6.11(b). An example of the general excitation containing both overlapping and modulating slow components is shown in Fig. 6.11(c).

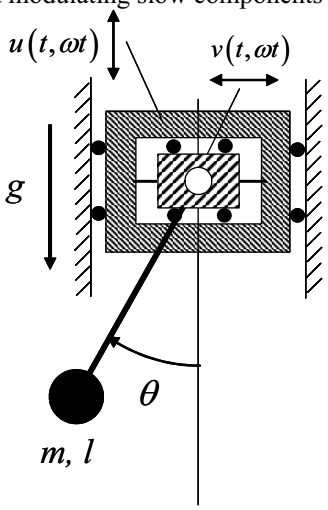

Fig. 6.10. The pendulum excited in two directions.

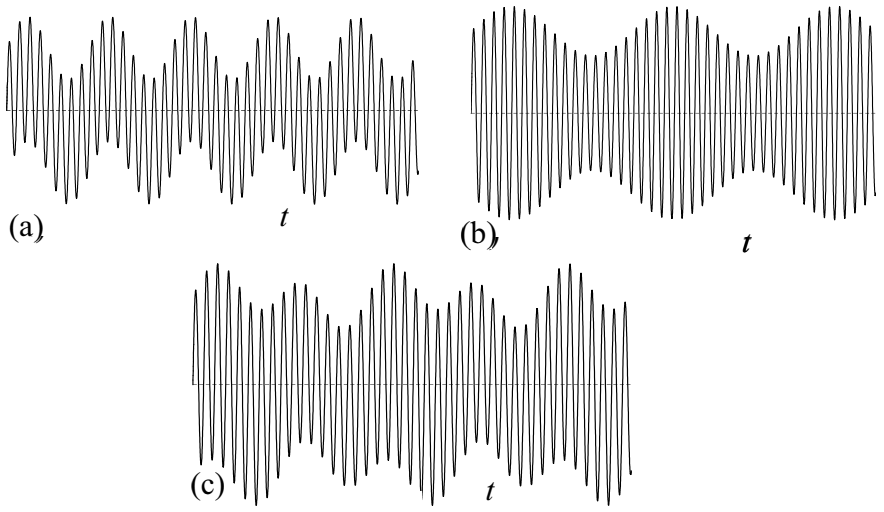

Fig. 6.11. Slowly overlapped and slowly modulated HF excitations

6.3.2 Obtaining Equations Governing the Slow Motions of the Pendulum

The equation governing motions of such a pendulum can be written in the undimensioned form as follows:

$$\ddot{\theta} + 2\beta\dot{\theta} + \sin\theta = -\ddot{u}\cos\theta - \ddot{v}\sin\theta \tag{6.32}$$

The following form can be presumed for the excitations:

$$\begin{aligned} u(t,\omega t) &= \bar{a}_{\perp}(t) + \tilde{a}_{\perp}(t)\sin\omega t \\ v(t,\omega t) &= \bar{a}_{\parallel}(t) + \tilde{a}_{\parallel}(t)\sin\omega t \end{aligned} \tag{6.33}$$

We will assume these given functions to have the following magnitude orders:

$$\begin{aligned} &\bar{a}_{\parallel} = O(1);\ \bar{a}_{\perp} = O(1);\ \tilde{a}_{\parallel}\omega = O(1);\ \tilde{a}_{\perp}\omega = O(1); \\ &\omega >> 1;\ \varepsilon = \omega^{-1} << 1 \end{aligned} \tag{6.34}$$

The assumption $\bar{a}_{\parallel} = \bar{a}_{\parallel}(t)$ means $\dot{\bar{a}}_{\parallel} = O(1); \ddot{\bar{a}}_{\parallel} = O(1)$. Similar estimations are valid for the functions $\tilde{a}_{\parallel}$, $\bar{a}_{\perp}$, $\tilde{a}_{\perp}$.

The following analysis is very similar to the previous section. The equation (6.32) can be transformed to a system of two first order differential equations (we convert to the fast time as the new independent variable and take (6.33) into account):

$$\begin{aligned}\theta' &= \varepsilon\rho \\ \rho' &= \tilde{a}_{\|}\omega\sin\tau\cos\theta - \left(\ddot{\tilde{a}}_{\|}\varepsilon + \dddot{\tilde{a}}_{\|}\varepsilon\sin\tau + 2\ddot{\tilde{a}}_{\|}\cos\tau\right)\cos\theta - 2\beta\varepsilon\rho \\ &+ \tilde{a}_{\perp}\omega\sin\tau\sin\theta - \left(\ddot{\tilde{a}}_{\perp}\varepsilon + \dddot{\tilde{a}}_{\perp}\varepsilon\sin\tau + 2\ddot{\tilde{a}}_{\perp}\cos\tau\right)\sin\theta - \varepsilon\sin\theta\end{aligned} \tag{6.35}$$

We can apply the following transformation based on the general solution to the unperturbed problem:

$$\rho = \sigma - \tilde{a}_{\|}\omega\cos\tau\cos\theta - \tilde{a}_{\perp}\omega\cos\tau\sin\theta \tag{6.36}$$

The result is a system in the standard form which can be averaged. The equations of the first order approximation are as follows:

$$\begin{aligned}\ddot{\theta} + 2\beta\dot{\theta} + \left(1 + \ddot{\tilde{a}}_{\|} + \frac{1}{2}\left(\tilde{a}_{\|}^{\,2} - \tilde{a}_{\perp}^{\,2}\right)\omega^2\cos\theta\right)\sin\theta \\ = -\ddot{\tilde{a}}_{\perp} - \frac{1}{2}\tilde{a}_{\|}\tilde{a}_{\perp}\omega^2\cos 2\theta\end{aligned} \tag{6.37}$$

We have returned here back to the original slow time. The equation (6.37) differs from the equation (6.28) discussed in the previous section through the fact that its right-hand side and the expression in brackets on the left-hand side depend explicitly on the slow time, i.e. they contain the slow result of the slowly overlapped and modulated HF excitation.

6.3.3 The Effect of the Overlapped slow Excitation

Consider the case without HF excitation first. In other words, assume $\tilde{a}_{\|} = \tilde{a}_{\perp} = 0$ and restrict the analysis to oscillations in the vicinity of the down-pointing equilibrium. Then the equation (6.37) can be linearized as follows:

$$\ddot{\theta} + 2\beta\dot{\theta} + \left(1 + \ddot{\tilde{a}}_{\|}\right)\theta = -\ddot{\tilde{a}}_{\perp} \tag{6.38}$$

This equation contains two excitation terms. The external excitation is determined by the horizontal oscillations of the suspension. The vertical oscillations of the base lead to the parametric excitation of the pendulum. In other words in the presence of the horizontal slow excitation the pendulum behaves as a nonlinear resonant oscillator. In the presence of the vertical slow excitation one can expect instabilities and parametric resonance.

This qualitative statement remains valid in the presence of the non modulated HF excitation. The only difference is that the resonance frequencies for the external excitation and the instability areas with respect to parametric excitation become related to the effective stiffness instead of the real one. The corresponding equation is

$$\ddot{\theta} + 2\beta\dot{\theta} + \left(1 + \ddot{\bar{a}}_{\parallel} + \frac{1}{2}\tilde{a}_{\parallel}^{\,2}\omega^2\right)\theta = -\ddot{\bar{a}}_{\perp} \tag{6.39}$$

If we consider for example the case of the vertical non modulated HF excitation ($\tilde{a}_{\parallel} = const$, $\tilde{a}_{\perp} = 0$) and presume that the overlapped horizontal excitation is a harmonic one ($\bar{a}_{\perp} = \bar{A}_{\perp}\cos\omega_{slow}t$) and the overlapped vertical excitation is equal to zero ($\ddot{\bar{a}}_{\parallel} = 0$), then the external resonance can be expected for the excitation's frequency

$$\omega_{slow} = \omega_{eff}^{0} = \sqrt{1 + \frac{1}{2}\tilde{a}_{\parallel}^{\,2}\omega^2} \neq 1 \tag{6.40}$$

If the overlapped horizontal excitation is equal to zero but the overlapped vertical excitation is a harmonic one, i.e. $\bar{a}_{\parallel} = \bar{A}_{\parallel}\cos\omega_{slow}t$, then slow parametric resonances can be expected at the frequencies determined according to the Strutt's diagram [119]. The relationships between the frequencies corresponding to the parametric resonances can be found as follows. The equation (6.39) has to be transformed to the Mathieu's form

$$\frac{d^2\theta}{dt_*^{\,2}} + \left(\delta_M + 2\varepsilon_M\cos 2t_*\right)\theta = 0 \tag{6.41}$$

It can be done if we neglect damping, introduce the new time and use the following notation:

$$t_* = \frac{1}{2}\omega_{slow}t;\ \ \delta_M = 4\frac{\omega_{eff}^2}{\omega_{slow}^2};\ \ \varepsilon_M = -2\bar{A}_{\parallel} \tag{6.42}$$

The roots of the parametric resonances correspond to

$$\delta_M = n^2 \Leftrightarrow \omega_{slow} = \frac{2\omega_{eff}}{n},\ n = 1, 2, \ldots \tag{6.43}$$

In other words the instability areas for parametric resonances are also shifted due to the HF excitation. This shift can be arbitrarily large if the HF excitation is

sufficiently strong. The resonances can be observed at unexpected excitation's frequencies as the consequence. Fig. 6.12 illustrates the described behavior.

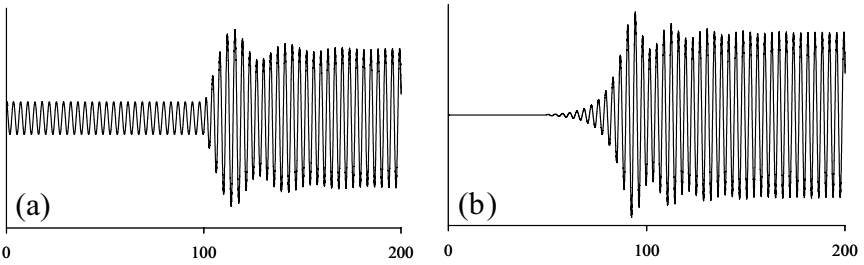

Fig. 6.12. Reaction of a pendulum to the suddenly switched on HF excitation: (a) – the slow external excitation according to the case (6.40); (b) – the slow parametric excitation according to the case (6.43)

In both cases the complete nonlinear pendulum was simulated. The pendulum is not HF excited at the beginning (Fig. 6.12 (a)). The pendulum oscillates with comparatively small amplitude because the excitation's frequency is far away from its natural frequency. The HF excitation is switched on at the time t=100. The effective frequency changes as a result and coincides with the excitation's frequency. The oscillations frequency remains constant but the amplitude increases significantly although the amplitude of the excitation did not changed. This effect can be interpreted as a shifted resonance. The numeric simulations were performed for the following parameter values:

$$\omega = 50;\ \tilde{a}_{\|} = 0.04;\ \overline{A}_{\perp} = 0.1;\quad \omega_{slow} = 1,732;\ \beta = 0.05\,.$$

In case (b) the slow excitation has the parametric nature. That's why the pendulum does not move at the beginning. As soon as the HF excitation is switched on (t=50) the effective frequency changes and the equilibrium becomes unstable with respect to slow parametric excitation. The amplitude increases exponentially until the nonlinearity limits it. The numeric simulations were preformed for the same parameter values as in the previous case but $\overline{A}_{\|} = 0.1;\ \omega_{slow} = 3.464;\ \overline{A}_{\perp} = 0$.

The whole situation may seem to be a mysterious one for an observer who does not notice the fast motions. He sees only the overlapped low excitation and knows that nothing has changed. Suddenly the amplitude start to increase or even the equilibrium becomes unstable. The only way to understand the situation is to broaden the frequency limits of the observer and notice the HF excitation. It becomes very visible if one measures the velocities (or accelerations) instead of the displacements. On the other hand the measuring system must be able to "see" the HF motions, i.e. its own resonance must be much higher than the HF excitation and the data processing system should not contain a low pass filter. These requirements can be easily fulfilled in mechanical engineering. In the electronics it might be much more difficult.

The same effect can be used in order to avoid the resonances. The stiffening or softening due to the controlled HF excitation can shift the effective frequency

away from the excitation's frequency and quench the dangerous or disturbing resonances. The particular advantage of this approach is that the HF excitation frequency can be so high that it would not disturb a normal customer. An example of quenching an external slow resonance by shifting of the effective natural frequency is displayed in Fig. 6.13.

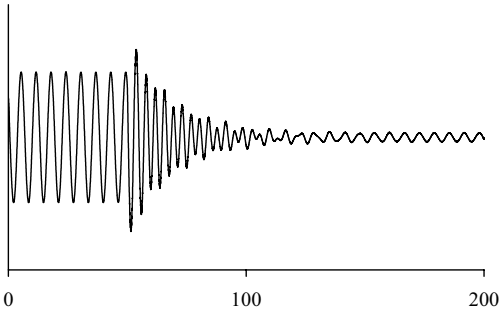

Fig. 6.13. The slow resonance quenched by the HF excitation switched on at t=50.

6.3.4 The Effect of the Slowly Modulated HF Excitation

The effect of a slowly modulated HF excitation is quite similar. Let us discuss the parametric case and consider the vertical excitation. The corresponding equation for the first order approximation is:

$$\ddot{\theta} + 2\beta\dot{\theta} + \left(1 + \ddot{\bar{a}}_{||} + \frac{1}{2}\tilde{a}_{||}^{\,2}\omega^2\right)\theta = 0 \tag{6.44}$$

The role of the modulated excitation is similar to that of the slow overlapped excitation. Assuming that there is no overlapped excitation and the modulation is a harmonic one, i.e. $\bar{a}_{||} = 0;\ \tilde{a}_{||} = \tilde{A}_{||}\cos\omega_{\text{mod}}t$, one obtains the Mathieu's equation with the following parameters

$$\delta = \frac{1 + \frac{1}{4}\tilde{A}_{||}^{\,2}\omega^2}{\omega_{\text{mod}}^2};\ \varepsilon = \frac{\tilde{A}_{||}^{\,2}\omega^2}{8\omega_{\text{mod}}^2} \tag{6.45}$$

The corresponding simulation results are shown in Fig. 6.14.

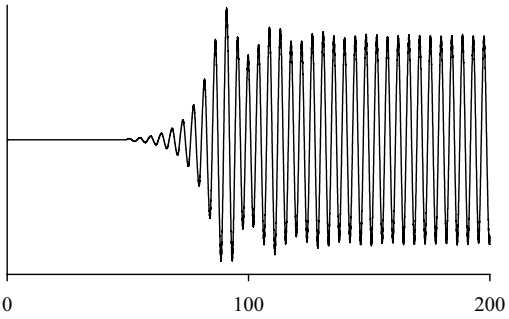

Fig. 6.14. The slow parametric resonance due to the slowly modulated HF excitation; the simulations were performed for the following parameters: $\omega = 50;\ \tilde{A}_{||} = 0.04;\ \omega_{\text{mod}} = 1.414;\ \beta = 0.05$.

The zoomed display of the modulated excitation and the pendulum's response are shown in Fig. 6.15.

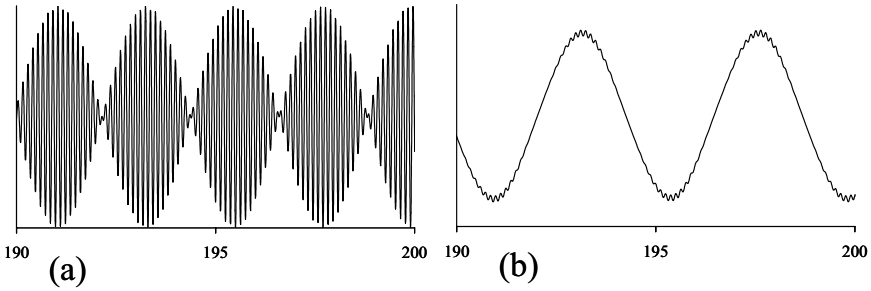

Fig. 6.15. (a) – The slowly modulated excitation; (b) – the slow parametric resonance

6.3.5 Using the Slowly Modulated HF Excitation in Order to Quench the Slow Excitation

The qualitative equivalence between the overlapped slow vibration and the slowly modulated HF excitation in the equation (6.44) is obvious. This fact indicates that these two influences can compensate each other. This compensation cannot be perfect, because the term $\ddot{\tilde{a}}_{||}$ usually does not contain a constant component. The term $\frac{1}{2}\tilde{a}_{||}^{\,2}\omega^2$ representing the effect of the slow modulation, however always contains the constant. Nevertheless the harmonically oscillating components can be always chosen in order to eliminate the excitation.

In our case of the slow harmonic parametric excitation the choice is very simple. The following condition must be fulfilled:

$$\begin{aligned}
&\ddot{\bar{a}}_{\|} = -\frac{1}{2}\tilde{a}_{\|}^{\,2}\omega^2 + const \\
&\bar{a}_{\|} = \bar{A}_{\|}\cos\omega_{slow}t;\ \tilde{a}_{\|} = \tilde{A}_{\|}\cos\omega_{\mathrm{mod}}t \quad \Rightarrow \\
&\omega_{slow} = 2\omega_{\mathrm{mod}} \\
&\bar{A}_{\|} = \frac{\tilde{A}_{\|}^{\,2}\omega^2}{16\omega_{\mathrm{mod}}^2}
\end{aligned} \tag{6.46}$$

The example of a numeric simulation illustrating this approach is displayed in Fig. 6.16. It is the direct prolongation of the simulation shown in Fig. 8.14. Beginning at the time point t=200 the slow overlapped excitation corresponding to (6.46) is switched on.

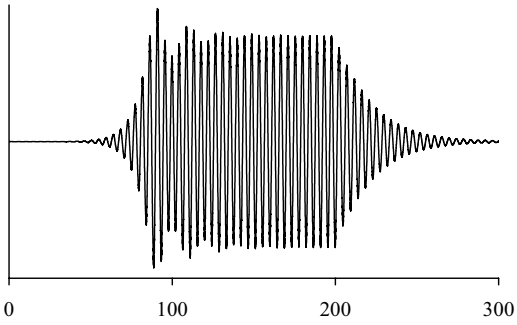

Fig. 6.16. The overlapped and modulated excitations compensate each other

The amplitude of the oscillations decreases as a result due to the normal damping. This approach can be used in both directions – to quench the slow parametric excitation by the applied modulated HF excitation and vice versa.

The same idea can be used to quench the external excitation too. The terms $\ddot{\bar{a}}_{\perp}$ and $\frac{1}{2}\tilde{a}_{\|}\tilde{a}_{\perp}\omega^2$ must compensate each other in this case. It means that modulated HF excitations in both directions (vertical and horizontal) are necessary in order to compensate the slow horizontal excitation. Another peculiarity in this case is the fact that the constant uncompensated component of the HF excitation will cause a constant bias of the pendulum from the down-pointing equilibrium.

This idea can also be used in order to quench or compensate the external and the parametric slow excitations by the appropriately modulated HF excitations simultaneously. The corresponding control strategies can be accomplished by the piezo-electric actuators.

6.4 The First Generalization and the Exceptional Role of the Terms Depending on the Velocity in Systems with HF Excitation

The approach we have used in all the examples can be easily generalized for a large class of discrete systems with strong HF excitation dependent only on the positional coordinates and time and independent from the velocities.

6.4.1 The Basic Equation of the Vibrational Mechanics

Consider a system described by n second order ordinary differential equations:

$$\ddot{x} = F\left(x, \dot{x}, t\right) + \omega\Phi\left(x, t, \omega t\right) \tag{6.47}$$

Here x is the vector of the positional coordinates, and the function Φ depends 2π -periodically on the argument ωt. It is important that the strong exciting term $\omega\Phi$ does not depend on the velocities $\dot{x}$. The fact that the function F does not depend on the fast time is not important but it is useful for the qualitative interpretation. The first step in the analysis if we are going to apply averaging is to convert to the fast time $\tau = \omega t$ as the new independent variable. The result is the following system:

$$\begin{aligned} x'' &= \varepsilon^2 F\left(x, \varepsilon^{-1}x', t\right) + \varepsilon\Phi\left(x, t, \tau\right) \\ t' &= \varepsilon \\ \varepsilon &= \omega^{-1} << 1 \end{aligned} \tag{6.48}$$

This system is in general unsuitable for the asymptotic analysis. The reason is the fact that the system's velocity written in terms of the fast time may be large. The variable $\varepsilon^{-1}x'$ can have the magnitude order ε^{-1} and the functional dependency of F from this argument is unknown. In other words this argument in F can generate terms of an arbitrary asymptotic order in our equations. The seemingly natural assumption that F must be a bounded function of all its arguments does not hold even for the elementary linear damping. The usual turbulent damping is proportional to $\dot{x}^2$. It would generate the large term $O\left(1\right)$ in the equations (6.48). The only case where the function F is usually bounded describes the dry friction. This case will be discussed in the next paragraph.

The problem was avoided in the previous examples because converting to a system of the first order differential equations we have introduced the velocity as follows:

$$x' = \varepsilon v$$
$$v' = \varepsilon F\left(x, v, t\right) + \Phi\left(x, t, \tau\right) \tag{6.49}$$
$$t' = \varepsilon$$

What does this assumption really mean? Returning back to the original slow time as the independent variable we could write $\dot{x} = v$. This relationship means that the coordinate and the velocity are the variables of the same magnitude order. This assumption obviously cannot be fulfilled for arbitrary motions or arbitrary initial conditions. Writing (6.49) we restrict our analysis to the motions with limited velocities $v = O\left(1\right) \Leftrightarrow x' = O\left(\varepsilon\right)$.

The same idea can be also formulated not for the solutions but for the right hand sides of the equations. We can require that F has to be bounded in the vicinity of the solution to the averaged system we investigate. The fulfillment of this requirement can be easily checked a posteriori. Fig. 6.17 shows the displacement and the velocity of the resonantly excited pendulum. It is clear that the requirements to the velocity are fulfilled in this case.

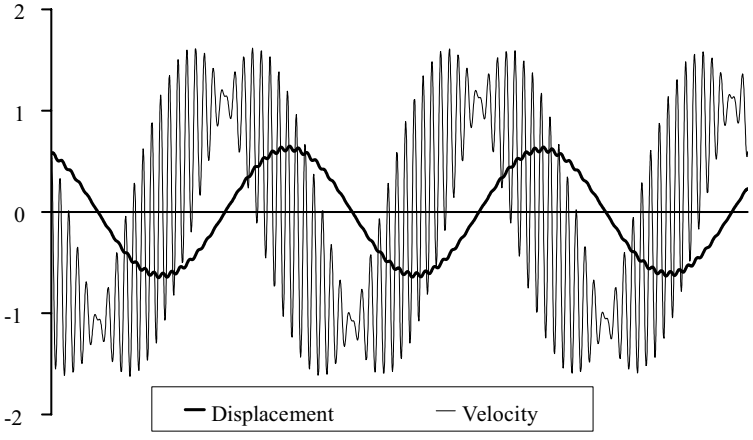

Fig. 6.17. Displacement and velocity of the resonantly oscillating HF excited pendulum.

This picture illustrates the most important property of all solutions discussed in this chapter. The motion of a strongly HF excited system is a superposition of a slow evolution (or slow oscillations) and small fast vibrations which are hardly visible if we consider (or measure) displacements. The velocity however is a superposition of the slow evolution and fast oscillations which have the same magnitude order. The fast oscillations of the velocity cannot be overseen in comparison to the averaged velocity.

This fact is also expressed in the procedure which we have used in order to transform the system (6.49) into the standard form suitable for the averaging.

This transformation obtained as the general solution to the unperturbed system is

$$\begin{aligned} v &= U\left(x,t,\tau\right)+u \\ \frac{\partial U}{\partial \tau} &= \Phi\left(x,t,\tau\right) \end{aligned} \tag{6.50}$$

This transformation means that the velocity is a sum of slow and fast oscillations of the same magnitude order. The function $U\left(x,t,\tau\right)$ describes the large fast velocity's oscillations. It is natural to require from this function, it has to be 2π -periodic with respect to the fast time and its average has to vanish.

$$U\left(x,t,\tau+2\pi\right)=U\left(x,t,\tau\right);\quad \left\langle U\right\rangle=\frac{1}{2\pi}\int_0^{2\pi}U\left(x,t,\tau\right)d\tau=0 \tag{6.51}$$

Applying this transformation to the system (6.49) one easily obtains:

$$\begin{aligned} x' &= \varepsilon\left(U\left(x,t,\tau\right)+u\right) \\ u' &= \varepsilon F\left(x,u+U,t\right)-\varepsilon\frac{\partial U}{\partial t}-\varepsilon\frac{\partial U}{\partial x}\left(u+U\right) \\ t' &= \varepsilon \end{aligned} \tag{6.52}$$

This is a system in the standard form. It can be averaged according to the usual procedure. Take the following identities into account:

$$\left\langle\frac{\partial U}{\partial t}\right\rangle=\frac{\partial}{\partial t}\left\langle U\right\rangle=0;\ \left\langle\frac{\partial U}{\partial x}\right\rangle=\frac{\partial}{\partial x}\left\langle U\right\rangle=0 \tag{6.53}$$

Then we immediately obtain the equations of the first order approximation (the averaged quantities have index 1):

$$\begin{aligned} x_1' &= \varepsilon u_1 \\ u_1' &= \varepsilon\left\langle F\left(x_1,u_1+U\left(x_1,t,\tau\right),t\right)\right\rangle-\varepsilon\left\langle\frac{\partial U}{\partial x_1}U\right\rangle \\ t' &= \varepsilon \end{aligned} \tag{6.54}$$

Sometimes another periodic function with vanishing average is used instead of U (*cf.* [20, 34, 35, 119]. We can introduce

$$\Psi\left(x_1,t,\tau\right):\frac{\partial\Psi}{\partial\tau}=U\left(x_1,t,\tau\right)\Leftrightarrow\frac{\partial^2\Psi}{\partial\tau^2}=\Phi\left(x_1,t,\tau\right)$$
$$\left\langle\Psi\right\rangle=0 \tag{6.55}$$

Notice that the following identity is fulfilled because all the functions are periodic:

$$\left\langle\frac{\partial U}{\partial x_1}U\right\rangle=\left\langle\frac{\partial^2\Psi}{\partial x_1\partial\tau}\frac{\partial\Psi}{\partial\tau}\right\rangle=-\left\langle\frac{\partial^3\Psi}{\partial x_1\partial^2\tau}\Psi\right\rangle=-\left\langle\frac{\partial\Phi}{\partial x_1}\Psi\right\rangle \tag{6.56}$$

Now we can rewrite (6.54) as the system of the second order differential equation with the slow time as the independent variable:

$$\ddot{x}_1=\left\langle F\left(x_1,\dot{x}_1+U\left(x_1,t,\tau\right),t\right)-\frac{\partial U}{\partial x_1}U\right\rangle \tag{6.57}$$

The equivalent form is:

$$\ddot{x}_1=\left\langle F\left(x_1,\dot{x}_1+\frac{\partial\Psi}{\partial\tau},t\right)+\frac{\partial\Phi}{\partial x_1}\Psi\right\rangle \tag{6.58}$$

This equation describes the slow motions in a strongly HF excited system and can be considered as the basic equation of the *vibrational mechanics* [20].

The form (6.57) is very similar to the original system (6.47) and shows clearly the principal effect of the HF excitation. It can be formulated in terms of the vibrational forces. The equations (6.57) or (6.58) can be rewritten as follows

$$\begin{aligned}
&\ddot{x}_1=F\left(x_1,\dot{x}_1,t\right)+V\left(x_1,\dot{x}_1,t\right)\\
&V\left(x_1,\dot{x}_1,t\right)=V_d\left(x_1,\dot{x}_1,t\right)+V_i\left(x_1,\dot{x}_1,t\right)\\
&V_d\left(x_1,\dot{x}_1,t\right)=-\left\langle\frac{\partial U}{\partial x_1}U\right\rangle\equiv\left\langle\frac{\partial\Phi}{\partial x_1}\Psi\right\rangle\\
&V_i\left(x_1,\dot{x}_1,t\right)=\left\langle F\left(x_1,\dot{x}_1+U\left(x_1,t,\tau\right),t\right)-F\left(x_1,\dot{x}_1,t\right)\right\rangle
\end{aligned} \tag{6.59}$$

Here we have explicitly separated the original slow forces F and the so called *effective vibrational forces* V contributing to the equations governing the slow motions of the system. An observer overlooking the fast motions (which are really small) could think the system moves as if there were some additional slow forces of the unknown physical origin. This misinterpretation would be based on the ignorance towards the small fast oscillations.

The vibrational forces can be split into two compounds. The first one is V_d. It can be characterized as the *direct vibrational force*. It was the origin for all the effects discussed above: stiffening, softening, biasing and shifted resonances. The second compound is V_i which can be called the *induced vibrational force*. It appears only if the real slow forces depend nonlinearly on the velocities. In the previous examples the only slow force depending on the velocities was the linear damping. The induced vibrational force did not appear in those cases, because the corresponding average vanishes:

$$\begin{aligned} &\left\langle \dot{x}_1 + U\left(x_1,t,\tau\right)\right\rangle = \dot{x}_1 + \left\langle U\left(x_1,t,\tau\right)\right\rangle = \dot{x}_1 \Rightarrow \\ &\left\langle F\left(x_1,t\right) + k\left(\dot{x}_1 + U\left(x_1,t,\tau\right)\right)\right\rangle - \left(F\left(x_1,t\right) + k\dot{x}_1\right) = 0 \end{aligned} \tag{6.60}$$

In the next section we will discuss how the HF excitation influences systems with dry friction which is an important example of nonlinear forces depending on the velocities. An additional advantage of the dry friction is the fact that it remains bounded even for large velocities. The induced vibrational forces are of the principal importance in these systems.

6.4.2 A Remark on the Exceptional Role of the Terms Depending on the Velocities

Before we continue with these examples, let us explain why we have excluded the velocities as an argument in the function Φ corresponding to the strong HF excitation. The equations (6.49) can be rewritten as follows in this general case:

$$\begin{aligned} x' &= \varepsilon v \\ v' &= \varepsilon F\left(x,v,t\right) + \Phi\left(x,v,t,\tau\right) \\ t' &= \varepsilon \end{aligned} \tag{6.61}$$

The corresponding unperturbed system is

$$\begin{aligned} x_0' &= 0 \Rightarrow x_0 = const \\ t_0' &= 0 \Rightarrow t_0 = const \\ v_0' &= \Phi\left(x_0,v_0,t_0,\tau\right) \end{aligned} \tag{6.62}$$

The last equation describes the fast motions of the system. Unfortunately its general solution cannot be written in the form (6.50)

$$v_0 = U\left(x_0,t_0,\tau,C\right) \neq U\left(x_0,t_0,\tau\right) + C \tag{6.63}$$

The corresponding transformation cannot be applied to our system. It means a different modified asymptotic approach is necessary in order to analyze a system

with a strong fast excitation depending on the velocities. A special case of these systems is discussed in section 6.6. It corresponds to the case of strongly dissipative systems. Systems with the HF excitation depending on the velocities will be discussed in the next chapter.

6.5 Smoothening of Dry Friction in Presence of HF Excitation. Quenching of the Friction Induced Oscillations

Several examples of systems with dry friction were discussed in Chapter 2. In this section we are going to consider these phenomena in presence of the HF excitation. The main result connected with the HF excitation in systems with dry friction is the "*smoothening*". This term describes the change in the system's behavior if one switch on the tangential HF excitation. The main point is: no sticking is possible in systems with dry friction if the HF excitation is strong enough. It is a usual reason for example for self-unscrewing bolts and nuts in vibrating machines. The same effect is often used in controlled mechanisms in order to reduce the undesired hysteresis disturbing the control system. In the specific language of the control specialists the HF excitation is called dither.

6.5.1 Smoothening of Dry Friction: A Simple Example

Let us consider an elementary example of a mass on a vibrating plane in order to demonstrate, how discontinuous dry friction can be smoothed through the external HF excitation (Fig. 6.18).

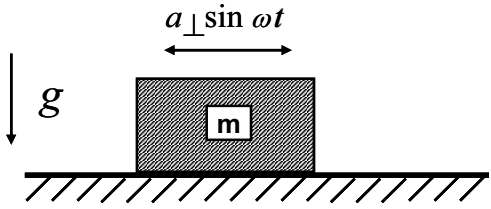

Fig. 6.18. A mass on the tangentially vibrating plane

Let us assume that the mass is pressed towards the plane by gravity. The friction between the mass and the plane is supposed to be a symmetric one. We consider the mass as a "point mass", i.e. all the effects connected with its "rocking" are neglected.

The equation of motion for this system can be written as follows (compare with the corresponding equations in Chapter 2):

$$m\ddot{x} = -mg\mu_{fr}\left(\dot{x}\right) + ma_{\perp}\omega^2 \sin\omega t$$

$$\mu_{fr}\left(\dot{x}\right) = \begin{cases} \mu \operatorname{sgn}\dot{x}, \ \text{if } \dot{x} \neq 0 \text{ or } |\mu| < \left|\dfrac{a_{\perp}\omega^2}{g}\sin\omega t\right| \\ \mu\dfrac{a_{\perp}\omega^2}{g}\sin\omega t, \ \text{if } \dot{x} = 0 \text{ and } |\mu| > \left|\dfrac{a_{\perp}\omega^2}{g}\sin\omega t\right| \end{cases} \tag{6.64}$$

Here x is the coordinate of the mass relative to the plane and $\dot{x}$ is the corresponding velocity determining the friction force. The friction coefficient is, as it was already mentioned, not a usual function but some kind of mapping being specific for dry friction. The last line in (6.64) expresses the possibility of sticking: if there is no relative motion between the mass and the plane and the external force is smaller that the maximum friction force. Then the friction force compensates the external force exactly and the mass remains motionless.

Introducing the fast time we can rewrite (6.64) as follows:

$$x'' = -\mu_{fr}\left(x'\right)g\varepsilon^2 + a_{\perp}\sin\tau, \ \ \varepsilon = \omega^{-1} \tag{6.65}$$

We suppose the excitation amplitude to be of the same magnitude order as our small parameter, i.e. $a_{\perp} = O\left(\varepsilon\right) \Leftrightarrow a_{\perp}\omega = O\left(1\right)$. The equation (6.65) can be rewritten as a system of two first order differential equations:

$$\begin{aligned} x' &= \varepsilon u \\ u' &= -\mu_{fr}\left(u\right)g\varepsilon + a_{\perp}\omega\sin\tau \end{aligned} \tag{6.66}$$

Using the obvious solution to the unperturbed system we can transform (6.66) to the standard form as follows:

$$\begin{aligned} u &= v - a_{\perp}\omega\cos\tau \\ &\begin{cases} x' = \varepsilon\left(v - a_{\perp}\omega\cos\tau\right) \\ v' = -\varepsilon g\mu_{fr}\left(v - a_{\perp}\omega\cos\tau\right) \end{cases} \end{aligned} \tag{6.67}$$

The system (6.67) seems to be in the standard form for averaging. However we must check if sticking is possible here or not. The relative velocity must be equal to zero for sticking. It means: if the mass has originally slipped it could stick if the following condition is fulfilled:

$$v = a_{\perp}\omega\cos\tau_{*} \tag{6.68}$$

But this condition is not sufficient for sticking. The additional condition is: the external force at this time point must be smaller than the maximal friction force, i.e.

$$\left|a_{\perp}\omega\sin\tau_{*}\right| < \mu g\varepsilon \tag{6.69}$$

We have terms of different magnitude order in this inequality. Its left hand side is large; its right hand side is small. This inequality can be fulfilled only if $\sin\tau_{*} = O(\varepsilon)$. It is a very short time interval. With the accuracy $O(\varepsilon)$ this time interval can be considered as an infinitesimally short one. It means that for the first order approximation no sticking is possible in a strongly HF excited system. The analysis must be extended if higher order approximations are of significant interest. Comparing (6.69) with (6.68) one can easily see that sticking could be possible only if the following condition if fulfilled for the velocity of the mass:

$$v = \pm a_{\perp}\omega \tag{6.70}$$

For arbitrary initial conditions it is obviously possible. The usual way however is to assume that $|v| < a_{\perp}\omega$ and check that for stationary solutions this condition is always fulfilled.

Summarizing, one can say that in systems with a strong HF excitation the complex friction law used in (6.64) can be replaced by a simple one:

$$\mu_{fr}(\dot{x}) = \mu\,\mathrm{sgn}\,\dot{x} \tag{6.71}$$

Inserting (6.71) into (6.67) one obtains:

$$\begin{cases} x' = \varepsilon\left(v - a_{\perp}\omega\cos\tau\right) \\ v' = -\varepsilon g\mu\,\mathrm{sgn}\left(v - a_{\perp}\omega\cos\tau\right) \end{cases} \tag{6.72}$$

The graph of the sign-function displayed in Fig. 6.19 explains the averaging.

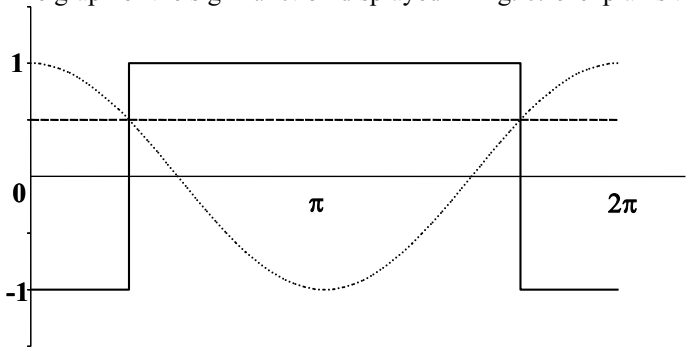

Fig. 6.19. The graph of the function $\mathrm{sgn}\left(a - b\cos\tau\right)$

$$\left\langle \operatorname{sgn}\left(\frac{v}{a_\perp\omega}-\cos\tau\right)\right\rangle = \begin{cases} 1, v > a_\perp\omega \\ 1-\frac{2}{\pi}\arccos\left(\frac{v}{a_\perp\omega}\right), -a_\perp\omega \le v \le a_\perp\omega \\ -1, v < -a_\perp\omega \end{cases} \quad (6.73)$$

The corresponding first order approximation to (6.72) can be written as follows:

$$\begin{cases} x' = \varepsilon v \\ v' = -\varepsilon g\mu\phi\left(\frac{v}{a_\perp\omega}\right) = -\varepsilon g\mu\left\langle \operatorname{sgn}\left(v - a_\perp\omega\cos\tau\right)\right\rangle \end{cases} \quad (6.74)$$

Function (6.73) describes the averaged or smoothened friction. Its graph is shown in Fig. 6.20.

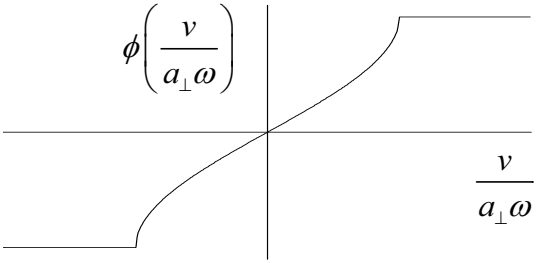

Fig. 6.20. The effective friction force in the presence of the strong HF excitation

This function is very interesting compared to the original dry friction. It remains limited if the relative velocity is sufficiently large, but the discontinuity at zero has disappeared. It is replaced by some kind of a continuous function and even more – the effective friction can be linearized for the sufficiently small relative velocities. The effective damping depends strongly on the intensity of the HF excitation. It decreases with the increasing excitation's intensity.

$$|v| << a_\perp\omega \Rightarrow \mu_{eff} \approx \frac{2}{\pi}\frac{v}{a_\perp\omega} \quad (6.75)$$

The only stationary solution to (6.74) is $v = 0$, because we have considered a system without any external force and with symmetric friction. If we add some force (for example for an inclined plane) we would obtain some non trivial velocity. The external force and the effective friction would balance each other in this case.

The same result can be easily obtained if we apply the equations of slow motions (6.59) to our system (6.65) rewritten as follows:

$$\ddot{x} = -\mu_{fr}(\dot{x})g + a_{\perp}\omega^2 \sin\omega t \tag{6.76}$$

Using the signs from the previous section we get the final equation directly:

$$\begin{aligned} &F = -\mu_{fr}(\dot{x})g;\ \Phi = a_{\perp}\omega\sin\tau;\ U = -a_{\perp}\omega\cos\tau \\ &V_d = 0;\ V_i = \mu_{fr}(\dot{x})g - \left\langle \mu_{fr}(\dot{x} - a_{\perp}\omega\cos\tau)g \right\rangle \Rightarrow \\ &\ddot{x}_1 = -\left\langle \mu_{fr}(\dot{x}_1 - a_{\perp}\omega\cos\tau)g \right\rangle \end{aligned} \tag{6.77}$$

6.5.2 Slow Translation of a Particle on the Elliptically Vibrating Plane

HF excitation can also set the particle in motion even if the friction law is completely symmetric. Let us consider the same mass on the plane but excite the plane elliptically (like in the section 6.2) in order to illustrate this statement (Fig. 6.21).

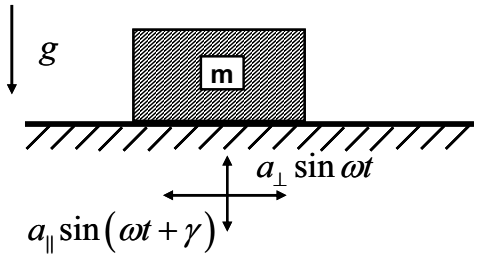

Fig. 6.21. A mass on the elliptically excited plane

The equations governing the motion of the mass can be written as follows:

$$\begin{aligned} &m\ddot{x} = -N\mu(\dot{x}) + ma_{\perp}\omega^2 \sin\omega t \\ &N = mg - ma_{\parallel}\omega^2 \sin(\omega t + \gamma) \end{aligned} \tag{6.78}$$

These equations express the fact that the friction force is proportional to the normal reaction force. In our case the normal force is influenced by the vertical acceleration of the plane. It is sensible to introduce the fast undimensioned time and two undimensioned parameters as follows:

$$k = g/a_{\parallel}\omega^2 = O(1);\ \lambda = \mu a_{\parallel}/a_{\perp}{}^2 = O(1) \tag{6.79}$$

We can rewrite (6.78) using these parameters and taking into account that sticking is not possible in presence of the strong HF excitation:

$$\ddot{x} = -\lambda a_{\perp}{}^2 \omega^2 \left(k - \sin\left(\omega t + \gamma\right)\right) \operatorname{sgn} \dot{x} + a_{\perp} \omega^2 \sin \omega t \tag{6.80}$$

We assume that the horizontal amplitude $a_{\perp}$ is the small parameter and transform this equation using the same approach as in the previous example:

$$\begin{aligned} F &= -\lambda a_{\perp}{}^2 \omega^2 \left(k - \sin\left(\tau + \gamma\right)\right) \operatorname{sgn} \dot{x};\ \Phi = a_{\perp} \omega \sin \tau \\ \ddot{x}_1 &= -\lambda a_{\perp}{}^2 \omega^2 \left\langle \left(k - \sin\left(\tau + \gamma\right)\right) \operatorname{sgn}\left(\dot{x} - a_{\perp} \omega \cos \tau\right) \right\rangle \end{aligned} \tag{6.81}$$

This system can be averaged. One can easily calculate the following integrals:

$$\begin{aligned} &\left\langle \sin \tau \operatorname{sgn}\left(v - \cos \tau\right) \right\rangle = 0 \\ &\left\langle \cos \tau \operatorname{sgn}\left(v - \cos \tau\right) \right\rangle = \psi\left(v\right) = \begin{cases} 0, |v| > 1 \\ -\frac{2}{\pi} \sin\left(\arccos v\right), |v| \le 1 \end{cases} \end{aligned} \tag{6.82}$$

The averaged equations can be written as follows:

$$\begin{aligned} x_1' &= a_{\perp} v_1 \\ v_1' &= -\lambda a_{\perp} \left\{ k \phi\left(v\right) - \sin \gamma \psi\left(v\right) \right\} \end{aligned} \tag{6.83}$$

It is obvious that a stationary solution is impossible if $|v_1| > 1$. If $|v_1| < 1$, then the stationary transportation velocity of the mass is governed by the following equation:

$$k\left(1 - \frac{2}{\pi} \arccos v_{st}\right) + \frac{2}{\pi} \sin \gamma \sin\left(\arccos v_{st}\right) = 0 \tag{6.84}$$

In general, this equation must be solved numerically. However in many cases the following approximation may be useful. We can introduce a new variable instead of the stationary velocity:

$$\begin{aligned} v_{st} &= \cos \varphi \\ \varphi - \frac{\sin \gamma}{k} \sin \varphi &= \frac{\pi}{2} \end{aligned} \tag{6.85}$$

The equation for φ can be solved if we assume that its solution is close to $\pi/2$. Then we can write:

$$\varphi = \frac{\pi}{2} + \alpha \Rightarrow \alpha - \frac{\sin\gamma}{k}\cos\alpha = 0$$
$$\alpha << 1 \Rightarrow \cos\alpha \approx 1 - \frac{\alpha^2}{2} \Rightarrow \alpha \approx \frac{\sin\gamma}{k} - \frac{1}{2}\left(\frac{\sin\gamma}{k}\right)^3 \tag{6.86}$$

Finally we obtain an approximate relationship for the velocity of stationary transportation of the mass on the elliptically HF excited plane:

$$\langle \dot{x} \rangle_{st} = -\frac{a_{\|} a_{\perp} \omega^3}{g} \sin\gamma \left(1 - \frac{2}{3}\left(\frac{a_{\|}\omega^2}{g}\right)^2 \sin^2\gamma\right) \tag{6.87}$$

It is interesting that the stationary transportation velocity does not depend on the friction coefficient. In Chapter 2 we have investigated a similar system. The transportation of the mass was caused by the asymmetry of the friction and the corresponding velocity was proportional to the asymmetry. In our case the reason for the transportation is the asymmetry of the excitation. It presses the mass towards the plane when it moves backwards and reduces the pressure in the phases, when the mass moves forwards. These influences result on the average in a steady state transportation. There are many machines based on this principle. First of all vibrating conveyors and sieves are used in the processing of bulk materials.

A comparison between the approximate solution (6.87) and the results of numeric simulations are shown in Fig. 6.22 for the following parameter values: $g = 10;\ \omega = 50;\ a_{\|} = 0.002;\ a_{\perp} = 0.01;\ \mu = 0.1$. The accuracy of the approximate result is acceptable.

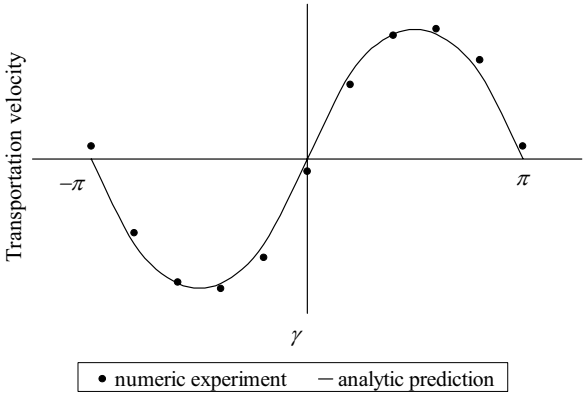

Fig. 6.22. Transportation of a particle on the elliptically vibrating plane: the averaged velocity

6.5.3 Quenching of the Self Excited Oscillations Caused by the Negative Friction Gradient

The ability of the HF excitation to smoothen the discontinuity in dry friction and to replace it by a continuous and monotonously increasing nonlinear damping can be used in order to quench the self excited oscillations caused for example by the negative friction gradient. We are going to illustrate this statement by an example which was discussed in [121]. Consider the classical "mass-on-moving-belt" as it was discussed in Chapter 2. How would the behavior change if we apply a strong tangential HF excitation? The corresponding system is shown in Fig. 6.23.

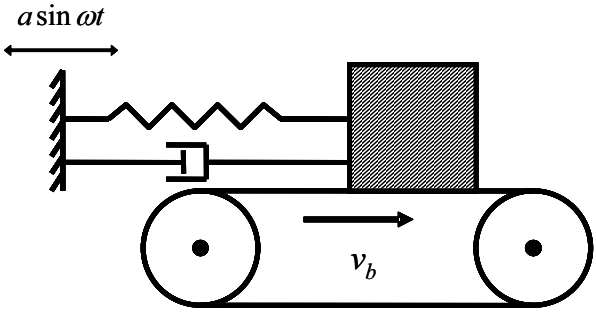

Fig. 6.23. The mass-on-moving-belt with a strong HF excitation

The equations of motion in the presence of the HF excitation can be written as follows (we use the model of friction which we have applied in the Chapter 2):

$$\begin{aligned} &\ddot{u}+2\beta\dot{u}+u=\mu\left(\dot{u}-v_b\right)+a\omega^2\sin\omega t \\ &\mu\left(\dot{u}-v_b\right)=\mu_s\,\mathrm{sgn}\left(\dot{u}-v_b\right)-k_1\left(\dot{u}-v_b\right)+k_3\left(\dot{u}-v_b\right)^3 \\ &\omega>>1,\ a\omega=O(1) \end{aligned} \tag{6.88}$$

The system (6.88) can be transformed to the basic form (6.47) if we use the following notation:

$$F=-u+h\left(\dot{u}\right);\ \Phi=a\omega\sin\tau \tag{6.89}$$

The equation governing the slow motions of the HF excited mass-on-moving-belt can be obtained immediately:

$$\begin{aligned} &U=-a\omega\cos\tau;\ V_d=0; \\ &\ddot{u}+u=\mu_{eff}\left(\dot{u}\right)=\left\langle\mu\left(\dot{u}-v_b-a\omega\cos\tau\right)\right\rangle \end{aligned} \tag{6.90}$$

The only difficulty is to calculate the average describing the effective friction characteristic. Taking (6.88) and (6.73) into account one can obtain the following result:

$$\mu_{eff}\left(\dot{u}\right)=\mu\left(\dot{u}-v_b\right)+\frac{3}{2}k_3a^2\omega^2\left(\dot{u}-v_b\right)+\begin{cases}\mu_s\left(\phi\left(\dfrac{\dot{u}-v_b}{a\omega}\right)-\operatorname{sgn}\left(\dot{u}-v_b\right)\right), & \text{for } \dot{u}-v_b\le a\omega\\ 0, & \text{for } \dot{u}-v_b\ge a\omega\end{cases} \tag{6.91}$$

The effective friction characteristic is displayed in Fig. 6.24.

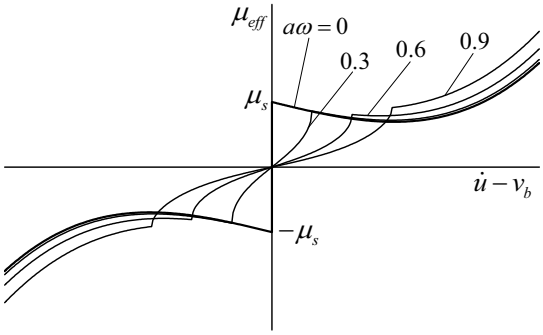

Fig. 6.24. Effective smoothed friction for different excitation's intensities

The result of the HF excitation is the smoothed friction characteristics. The effective friction is an increasing function of the relative velocity in the vicinity of zero, i.e. no self excitation is possible if the relative velocity is sufficiently small. On the other hand the minimum of the effective friction moves towards smaller relative velocities. The self-excitation remains possible only in the reduced velocity range

$$a\omega<\dot{u}-v_b<v_m^{eff}=\sqrt{\frac{k_1}{3k_3}-\frac{1}{2}a^2\omega^2} \tag{6.92}$$

Here v_m^{eff} is the relative velocity corresponding to the minimum of the effective friction coefficient. The self-excitation is impossible if

$$a\omega\ge v_m^{eff}\Leftrightarrow a\omega\ge\sqrt{\frac{2k_1}{9k_3}} \tag{6.93}$$

The amplitude of the corresponding limit cycle can be significantly reduced, even if this inequality is not fulfilled and the self-excitation takes place. Figure 6.25 shows a representative simulation with continuously increasing amplitude of the HF excitation ($\omega = 100$).

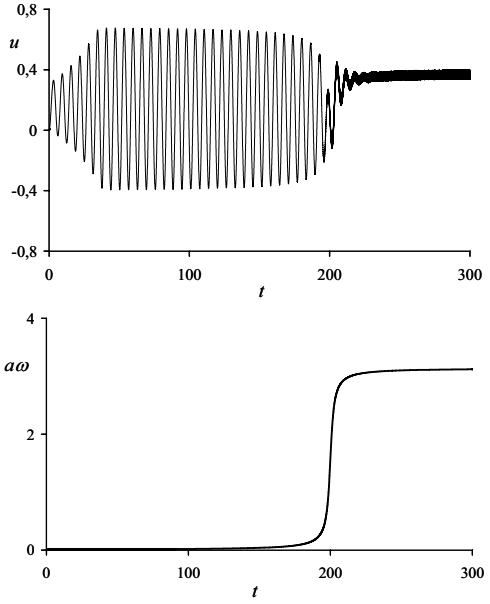

Fig. 6.25. Quenching of the friction induced oscillations due to the strong HF excitation

The excitation intensity is negligible at the beginning. The stable limit cycle establishes itself after a short transient process. The increasing excitation intensity at the time point about t=200 quenches the self-excited oscillations completely. The thick line at the end of the plot shows the small oscillations corresponding to the HF excitation. Figure 6.26 shows how the stationary oscillation's amplitude depends on the belt's velocity for different intensities of the HF excitation. The simulations were performed for $\mu_s = 0.2;\ k_1 = 0.1;\ k_3 = 0.04$.

The increasing excitation's intensity reduces both the instability area and the amplitude of the stable limit cycle. The whole velocities range remains stable for $a\omega = 0.9$.

The described effect seems to be an efficient way if it is necessary to avoid the self-excited oscillations induced by the negative friction gradient. However it is coupled with a principle disadvantage which significantly reduces the application's area. The described quenching mechanism is based on the smoothening effect of the HF excitation, i.e. on the transformation of the dry friction in an effective nonlinear damping. The main feature of the dry friction is sticking. This feature is not possible in the strongly excited system.

This quenching mechanism is not useful as the consequence in all the systems in which the sticking is necessary.

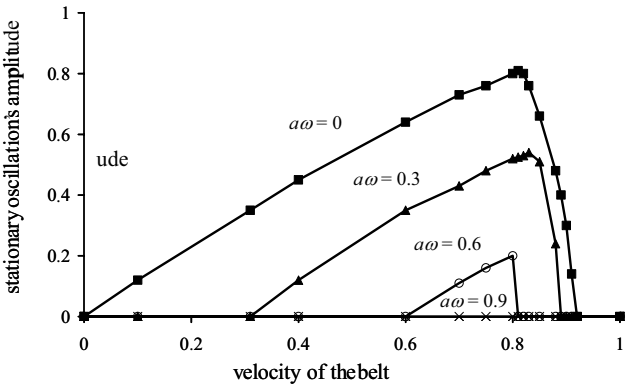

Fig. 6.26. Stationary amplitude of the limit cycle for different excitation intensities

Brakes and dry friction clutches are the typical examples for the systems in which this mechanism can be hardly used because their functionality is based on the sticking and the disturbing oscillations or noise occur at very small relative velocities just before sticking.

The HF excitation can be an interesting technical alternative to lubrication in the applications for which the sticking does not play such an important role (like sliding elements in control units, production machines) and the dry friction is only an unavoidable companion.

6.6. On the Misbehavior of the "Optimally" Controlled Pendulum under the Influence of the HF Excitation

The last example in this chapter deals with the behavior of a control system under the influence of the HF excitation. This problem was investigated by the author together with J. J. Thomsen.

6.6.1 Description of the Problem, Equations Governing the Mechanical Subsystem

Figure 6.27 shows the standard system normally used to test control algorithms. It contains a cart used to balance a pendulum in the up-pointing position against the gravitation force.

The system state can be described through two degrees of freedom – the position of the cart S and pendulum angle θ as observed from the rigid platform. The cart has the mass M and the linear damping coefficient d. The pendulum has the mass m and the torsion inertia J about its center of gravity at distance L from the loss free hinge. The system's reaction to perturbations is governed by a feedback control force $U = U\left(S, \dot{S}, \theta, \dot{\theta}\right)$. The rigid platform can be excited kinematically relative to the fixed inertial frame.

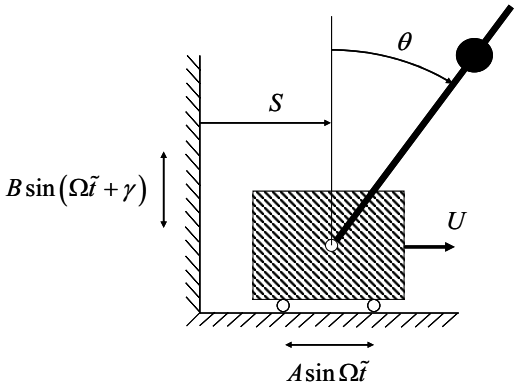

Fig. 6.27. Inverted pendulum balanced by a moving cart

The system's motion is governed by the following equations:

$$\begin{aligned} &\ddot{s} + 2\beta\dot{s} + \alpha\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) = u + a\omega^2\sin\omega t \\ &\ddot{\theta} - \left(1 - b\omega^2\sin\left(\omega t + \gamma\right)\right)\sin\theta + \ddot{s}\cos\theta = a\omega^2\sin\omega t\cos\theta \end{aligned} \tag{6.94}$$

The following undimensioned variables and parameters are introduced in these equations (*cf.* Fig. 6.27):

$$\begin{aligned} &\omega_0 = \sqrt{\frac{g}{l}};\ t = \omega_0\tilde{t};\ l = L\left(1 + \frac{J}{mL^2}\right);\ u = \frac{U}{(M+m)\omega_0{}^2 l}; \\ &s = \frac{S}{l};\ a = \frac{A}{l};\ b = \frac{B}{l};\ \alpha = \frac{m}{M+m}\frac{L}{l} < 1;\ \omega = \frac{\Omega}{\omega_0} \\ &\dot{(\ )} = \frac{d(\ )}{dt} \end{aligned} \tag{6.95}$$

6.6.2 The Optimal State-Feedback Control

The control u in (6.94) is intended to keep the system at zero state $z = \{s;\ \dot{s};\ \theta;\ \dot{\theta}\} = 0$, and return it back in case of any perturbations due to initial conditions or external forces. It is designed as an optimal linear regulator, basing its actions on a linearized model of the unforced system to be controlled, and on measurements – assumed here to be error- and noise-free – of instantaneous values of the state z, i.e., $u = u(z)$. The linearized model of the unforced system can be obtained from (6.94) if we ignore the excitation terms, i.e. $a = 0;\ b = 0$, and linearize the equations for small values of θ and $\dot{\theta}$ in order to obtain a system of linear first order differential equations in the state space form:

$$\dot{z} = \mathbf{A}z + \mathbf{B}u$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \dfrac{-2\beta}{1-\alpha} & \dfrac{-\alpha}{1-\alpha} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{2\beta}{1-\alpha} & \dfrac{1}{1-\alpha} & 0 \end{bmatrix};\quad \mathbf{B} = \begin{bmatrix} 0 \\ \dfrac{1}{1-\alpha} \\ 0 \\ \dfrac{-1}{1-\alpha} \end{bmatrix} \tag{6.96}$$

The control law $u(z)$ for the optimal linear state-feedback controller is:

$$u = -k^T z \tag{6.97}$$

Here the optimal gain vector k is chosen in order to minimize a weighted sum of integrated control error and control effort:

$$J = \int_0^\infty \left(z^T \mathbf{Q} z + R u^2(z) \right) dt \to \min \tag{6.98}$$

Here $\mathbf{Q}$ and R are positive definite weighting factors.These factors are chosen to reflect a proper tradeoff between the conflicting requirements of quickly reducing the set point error to a small value (large $\mathbf{Q}/R$), without expending excessively large control power (small $\mathbf{Q}/R$).

The optimal solution for k can be found in many texts on control theory (see for example [67]):

$$k = R^{-1}\mathbf{B}^T\mathbf{S} \tag{6.99}$$

Matrix $\mathbf{S}$ here is the solution to the algebraic Riccati's equation:

$$\mathbf{A}^T\mathbf{S} + \mathbf{S}\mathbf{A} - R^{-1}\mathbf{S}\mathbf{B}\mathbf{B}^T\mathbf{S} + \mathbf{Q} = 0 \tag{6.100}$$

For given values of $\mathbf{A}, \mathbf{B}, \mathbf{Q}, R$, the Riccati matrix $\mathbf{S}$ can be easily computed, e.g., using the built-function LQR in MATLAB [14] .

Thus, for given damping and mass ratio of the physical system, **A** and **B** is computed using (6.96), and with chosen $\mathbf{Q}$ and R, the Riccati matrix is given by the solution to (6.100). Then k is calculated using (6.99), and the control force u can be calculated using (6.97) for any state z.

The optimal control law can be represented explicitly as follows:

$$u = -k_1 s - k_2 \dot{s} - k_3 \theta - k_4 \dot{\theta} \tag{6.101}$$

Thus the controller simply has the effect of adding positive linear stiffness and damping to the uncontrolled system. The control coefficients are large compared to the parameters of the mechanical part of the system (6.94) and guarantee the stability of the set point $z = 0$ with respect to small perturbations.

6.6.3 System's Behavior in Presence of the Strong HF Excitation: Numeric Results

The question then is how the controlled system behaves in the presence of small but rapid oscillations of the supporting platform, i.e. if $a \neq 0$ and $b \neq 0$ in (6.94). It was already discussed above that the elliptic HF-excitation can lean the pendulum from its stable equilibrium position. The control system however is not limited in its power. Consequently, one might expect that the performance of the controller would be only slightly affected by the HF-excitation. The set point $z = 0$ would still remain stable, though with a small overlay of high-frequency (HF) oscillations corresponding to the excitation, or another stable equilibrium point would appear deviating slightly from $z = 0$. This deviation should decrease with the increasing control effort.

Numerical simulations however do not confirm these expectations (see Fig. 6.28). The control system works perfectly if only vertical or only horizontal excitation is present. It stabilizes the pendulum in the up-pointing position and moves the cart to the zero point. The motion of the pendulum and the cart is overlaid by small HF-oscillations but on the average the control system fulfills its objective. The situation changes completely if vertical and horizontal oscillations are applied simultaneously. The control system still stabilizes the cart and the pendulum, but the final averaged cart position becomes wrong and the pendulum is stabilized with some permanent tilt (see Fig. 6.29). An attempt to reduce the permanent error through increased control effort does not help at all. The amplified control system amplifies the misbehavior. This is the effect we are going to discuss in this section.

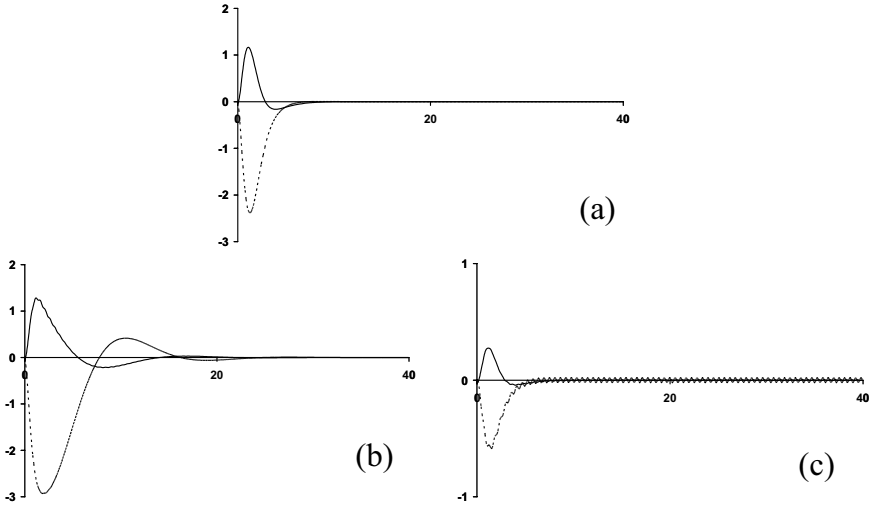

Fig. 6.28. Numeric simulations of the cart and the pendulum; (a) – the controlled system without HF-excitation; (b) – the system with the horizontal HF-excitation; (c) – the system with vertical HF-excitation; the dotted line shows the cart's position; the solid line shows the pendulum's tilt

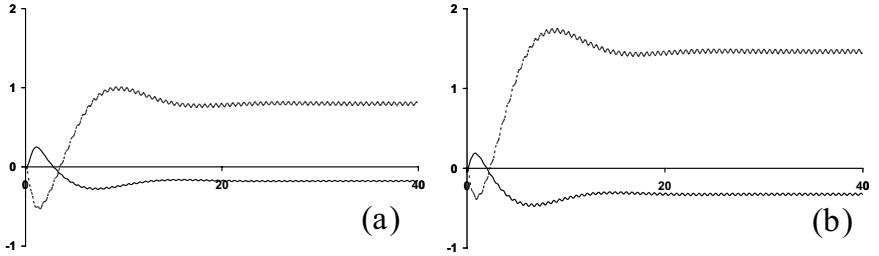

Fig. 6.29. Numeric simulations of the cart and the pendulum in the presence of the overlaid vertical and horizontal HF-excitations; (a) – the LQR optimal controlled system; (b) – the same system with amplified control effort. The dotted line shows the cart's position; the solid line shows the pendulum's tilt

6.6.4 Transformation of the System to the Form Suitable for Averaging

The objective of the following discussion is to explain the strange effect and to obtain the approximate analytic predictions for the stationary cart's position and pendulum's tilt. The system's motion is governed by the following equations:

$$\begin{aligned}\ddot{s} &= -k_1 s - k_2 \dot{s} - k_3 \theta - k_4 \dot{\theta} + a\omega^2 \sin \omega t \\ \ddot{\theta} &= \left(k_1 s + k_2 \dot{s} + k_3 \theta + k_4 \dot{\theta}\right)\cos\theta + \left(1 - b\omega^2 \sin\left(\omega t + \gamma\right)\right)\sin\theta\end{aligned} \tag{6.102}$$

The described effect remains if one simulates the linearized system instead of (6.102). That's why we are going to discuss the linearized variant too:

$$\begin{aligned}\ddot{s} &= -k_1 s - k_2 \dot{s} - k_3 \theta - k_4 \dot{\theta} + a\omega^2 \sin \omega t \\ \ddot{\theta} &= k_1 s + k_2 \dot{s} + k_3 \theta + k_4 \dot{\theta} + \left(1 - b\omega^2 \sin\left(\omega t + \gamma\right)\right)\theta\end{aligned} \tag{6.103}$$

Let us notice one important difference between the controlled system (6.103) and usual mechanical systems. This difference is the high damping level which is typical for any optimal control law. The reason for the strong damping is the objective to avoid the unnecessary oscillations and to lead the controlled system to its final position as fast as possible. This fact is the main difference (in the frame of this chapter) between the controlled systems and pure mechanical systems which are usually weakly damped.

Let us see to which form this system can be transformed and what kind of averaging would be appropriate for this form. We introduce firstly the fast time and the corresponding small parameter:

$$\begin{aligned}&\tau = \omega t;\ (\ )' = \frac{d(\ )}{dt};\ \varepsilon = \frac{1}{\omega} << 1; \\ &\overline{a} = a\omega = O(1);\ \overline{b} = b\omega = O(1) \\ &s' = \varepsilon p \\ &\theta' = \varepsilon q \\ &p' = -k_2 p - k_4 q + \overline{a}\sin\tau - \varepsilon\left(k_1 s + k_3 \theta\right) \\ &q' = k_2 p + k_4 q - \overline{b}\,\theta \sin\left(\tau + \gamma\right) + \varepsilon\left(k_1 s + \left(k_3 + 1\right)\theta\right)\end{aligned} \tag{6.104}$$

It is easy to notice, that the first two variables are slow, but the third and the forth variables are fast. The peculiarity of this system is that it is possible to introduce the third slow variable. The reason is the very special structure of the control applied only to the cart. If we add the third equation to the forth one, we obtain in the unperturbed case $\varepsilon = 0$ a simple relationship:

$$\begin{aligned}&p_0' + q_0' = \overline{a}\sin\tau - \overline{b}\,\theta_0 \sin\left(\tau + \gamma\right) \\ &\theta_0' = 0\end{aligned} \tag{6.105}$$

So it is sensible to replace for example p through a new slow variable x:

$$x = p + q + \bar{a}\cos\tau - \bar{b}\theta\cos(\tau+\gamma) \tag{6.106}$$

The corresponding transformed form for (6.104) is:

$$\begin{aligned}
s' &= \varepsilon\left(x - q - \bar{a}\cos\tau + \bar{b}\theta\cos(\tau+\gamma)\right)\\
\theta' &= \varepsilon q\\
x' &= \varepsilon\left(\theta - \bar{b}q\cos(\tau+\gamma)\right)\\
q' &= -(k_2-k_4)q + k_2\left(x - \bar{a}\cos\tau + \bar{b}\theta\cos(\tau+\gamma)\right)\\
&\quad -\bar{b}\theta\sin(\omega t+\gamma) + \varepsilon\left(k_1 s + (k_3+1)\theta\right)
\end{aligned} \tag{6.107}$$

The first three equations now have the standard form, but the last one contains both large oscillating terms, and the large linear term, expressing strong damping due to optimal control. The oscillating terms can be eliminated as follows. Let us consider the unperturbed system to (6.107) and try to find its particular periodic solution:

$$\begin{aligned}
q_0 &= \lambda_1 x_0 + \lambda_2\sin\tau + \lambda_3\cos\\
&\quad + \theta_0\left(\lambda_4\sin(\tau+\gamma) + \lambda_5\cos(\tau+\gamma)\right)
\end{aligned} \tag{6.108}$$

Inserting (6.108) into (6.107) and balancing terms with the same trigonometric functions one obtains the following expressions for the coefficients:

$$\begin{aligned}
\lambda_1 &= \frac{k_2}{k_2-k_4}\\
\lambda_2 &= -\frac{k_2\bar{a}}{1+(k_2-k_4)^2};\quad \lambda_3 = (k_2-k_4)\lambda_2 = -\frac{k_2(k_2-k_4)\bar{a}}{1+(k_2-k_4)^2}\\
\lambda_4 &= \frac{k_4\bar{b}}{1+(k_2-k_4)^2};\quad \lambda_5 = (k_2-k_4)\lambda_4 + \bar{b} = \frac{k_2(k_2-k_4)+1}{1+(k_2-k_4)^2}\bar{b}
\end{aligned} \tag{6.109}$$

Now we can apply to (6.107) the following transformation:

$$\begin{aligned}
q &= \lambda_1 x + y + \tilde{q}\\
\tilde{q} &= \lambda_2\sin\tau + \lambda_3\cos\tau + \lambda_4\theta\sin(\tau+\gamma) + \lambda_5\theta\cos(\tau+\gamma)
\end{aligned} \tag{6.110}$$

In the new variables system (6.107) takes it final form:

$$
\begin{aligned}
s' &= \varepsilon\left(x - \lambda_1 x - y - \tilde{q} - \bar{a}\cos\tau + \bar{b}\,\theta\cos\left(\tau+\gamma\right)\right)\\
\theta' &= \varepsilon\left(\lambda_1 x + y + \tilde{q}\right)\\
x' &= \varepsilon\left(\theta - \bar{b}\left(\lambda_1 x + y + \tilde{q}\right)\cos\left(\tau+\gamma\right)\right)\\
y' &= -\left(k_2 - k_4\right)y + \varepsilon\left(k_1 s + \left(k_3+1\right)\theta\right)\\
&\quad - \varepsilon\left(\lambda_1 x + y + \tilde{q}\right)\left(\lambda_4\sin\left(\tau+\gamma\right) + \lambda_5\cos\left(\tau+\gamma\right)\right)
\end{aligned}
\tag{6.111}
$$

This is the form, which is typical for systems with strong damping. The corresponding averaging procedure was considered in Chapter 4. It can be applied directly to this system if the coefficient $\left(k_2 - k_4\right)$ is positive. It is normally the case, because it is also the necessary condition for the stability of the control algorithm.

6.6.5 The First Order Approximation; the Stationary Pendulum's Tilt

Two important properties of this system can be noticed here already. The variable y is a slave due to the strong damping coefficients k_2 and k_4. It means that at least in the first order approximation the last equation in (6.111) will not give any additional information except the obvious result: y is small. The second important fact is that the only nontrivial term comes from the averaging of the product $\tilde{q}\cos\left(\tau+\gamma\right)$. This term describes the reaction of the control system on the HF-excitation. Relationships (6.109) show that this reaction is determined by the strong damping terms in the control system.

Let us start with the first order approximation. The variables in the first approximation are indicated with index 1.

$$
\begin{aligned}
s_1' &= \varepsilon\left(1 - \lambda_1\right)x_1\\
\theta_1' &= \varepsilon\lambda_1 x\\
x_1' &= \varepsilon\left(\theta_1 - \bar{b}\left\langle\tilde{q}\cos\left(\tau+\gamma\right)\right\rangle\right)\\
y_1' &= -\left(k_2 - k_4\right)y_1
\end{aligned}
\tag{6.112}
$$

The only average we have to calculate is

$$
\left\langle\tilde{q}\cos\left(\tau+\gamma\right)\right\rangle = -\frac{1}{2}\lambda_2\sin\gamma + \frac{1}{2}\lambda_3\cos\gamma + \frac{1}{2}\lambda_5\theta_1 \tag{6.113}
$$

Substituting (6.113) into (6.112) finally we get equations of the first order approximation:

$$\begin{aligned} s_1' &= \varepsilon\left(1-\lambda_1\right)x_1 \\ \theta_1' &= \varepsilon\lambda_1 x_1 \\ x_1' &= \varepsilon\left(\theta_1 + \frac{\bar{b}}{2}\left(\lambda_2\sin\gamma - \lambda_3\cos\gamma - \lambda_5\theta_1\right)\right) \\ y_1' &= -\left(k_2 - k_4\right)y_1 \end{aligned} \tag{6.114}$$

Stationary solution to this system can be found if we put its right hand sides to zero. We get the stationary value for the pendulum's tilt

$$\begin{aligned} \theta_{1\infty} &= -\bar{b}\,\frac{\lambda_2\sin\gamma - \lambda_3\cos\gamma}{2-\bar{b}\,\lambda_5} \\ &= \frac{k_2\bar{a}\bar{b}\left(\sin\gamma - \left(k_2-k_4\right)\cos\gamma\right)}{2\left(1+\left(k_2-k_4\right)^2\right) - \bar{b}^2\left(1+k_2\left(k_2-k_4\right)\right)} \end{aligned} \tag{6.115}$$

Unfortunately we do not get any information about the stationary cart position in the first approximation. That's why we need the second one. The corresponding procedure was formulated in Chapter 4. Let us shortly repeat these results.

6.6.6 The Second Order Approximation; the Stationary Position of the Cart

Consider a system containing both slow master variables and strongly damped slaves:

$$\begin{aligned} x' &= \varepsilon X\left(x,y,t\right) \\ y' &= -ky + \varepsilon Y\left(x,y,t\right) \\ x\left(0\right) &= x_0;\, y\left(0\right) = y_0 \end{aligned} \tag{6.116}$$

The second order approximation to this system is given by the following relationships:

$$\begin{aligned}
&\xi' = \varepsilon\Xi_1(\xi) + \varepsilon^2\Xi_2(\xi) + O(\varepsilon^2)\\
&\eta' = -k\eta + \varepsilon Y(\xi,0,t) + O(\varepsilon^2)\\
&\xi(0) = x_0; \eta(0) = y_0\\
&\Xi_1(\xi) = \langle X(\xi,0,t)\rangle; u_1 = \int_0^t \left[X(\xi,0,\tau) - \Xi_1(\xi)\right] d\tau\\
&\Xi_2(\xi) = \left\langle \left.\frac{\partial X}{\partial x}\right|_{y=0} u_1 + \frac{1}{k}\left.\frac{\partial X}{\partial y}\right|_{y=0} Y(\xi,0,t)\right\rangle
\end{aligned} \tag{6.117}$$

Now we are going to apply this approach to the system (6.111). The first step is to calculate the functions u :

$$\begin{aligned}
u_s &= \lambda_2\cos\tau - (\lambda_3 + \bar{a})\sin\tau + \lambda_4\theta\cos(\tau+\gamma)\\
&\quad -(\lambda_5 - \bar{b})\theta\sin(\tau+\gamma)\\
u_\theta &= -\lambda_2\cos\tau + \lambda_3\sin\tau - \lambda_4\theta\cos(\tau+\gamma) + \lambda_5\theta\sin(\tau+\gamma)\\
u_x &= -\bar{b}\lambda_1 x\sin(\tau+\gamma) + \frac{1}{4}\bar{b}\lambda_2\cos(2\tau+\gamma) - \frac{1}{4}\bar{b}\lambda_3\sin(2\tau+\gamma)\\
&\quad + \frac{1}{4}\bar{b}\lambda_4\theta\cos(2\tau+2\gamma) - \frac{1}{4}\bar{b}\lambda_5\theta\sin(2\tau+2\gamma)
\end{aligned} \tag{6.118}$$

Besides that we need the following derivatives:

$$\begin{aligned}
&\frac{\partial X_s}{\partial s} = 0; \frac{\partial X_\theta}{\partial s} = 0; \frac{\partial X_x}{\partial s} = 0\\
&\frac{\partial X_s}{\partial x} = 1-\lambda_1; \frac{\partial X_\theta}{\partial x} = \lambda_1; \frac{\partial X_x}{\partial x} = -\lambda_1\bar{b}\cos(\tau+\gamma)
\end{aligned} \tag{6.119}$$

$$\begin{aligned}
&\frac{\partial X_s}{\partial\theta} = -\lambda_4\sin(\tau+\gamma) - (\lambda_5 - \bar{b})\cos(\tau+\gamma)\\
&\frac{\partial X_\theta}{\partial\theta} = \lambda_4\sin(\tau+\gamma) + \lambda_5\cos(\tau+\gamma)\\
&\frac{\partial X_\theta}{\partial\theta} = 1 - \frac{1}{2}\bar{b}\lambda_5 - \frac{1}{2}\bar{b}\lambda_4\sin(2\tau+2\gamma) - \frac{1}{2}\bar{b}\lambda_5\cos(2\tau+2\gamma)
\end{aligned} \tag{6.120}$$

$$\frac{\partial X_s}{\partial y}=-1;\ \frac{\partial X_\theta}{\partial y}=1;\ \frac{\partial X_x}{\partial y}=-\bar{b}\cos(\tau+\gamma) \tag{6.121}$$

The function Y is given as follows:

$$Y=k_1 s+(k_3+1)\theta-(\lambda_1 x+\tilde{q})\left(\lambda_4\sin(\tau+\gamma)+\lambda_5\cos(\tau+\gamma)\right) \tag{6.122}$$

Now we can calculate the functions Ξ_{2s} and $\Xi_{2\theta}$:

$$\begin{aligned}\Xi_{2s}&=\left\langle\frac{\partial X_s}{\partial s}u_s+\frac{\partial X_s}{\partial\theta}u_\theta+\frac{\partial X_s}{\partial x}u_x+\frac{1}{k_2-k_4}\frac{\partial X_s}{\partial y}Y\right\rangle\\ \Xi_{2\theta}&=\left\langle\frac{\partial X_\theta}{\partial s}u_s+\frac{\partial X_\theta}{\partial\theta}u_\theta+\frac{\partial X_\theta}{\partial x}u_x+\frac{1}{k_2-k_4}\frac{\partial X_\theta}{\partial y}Y\right\rangle\end{aligned} \tag{6.123}$$

The result is

$$\begin{aligned}\Xi_{2s}&=\frac{1}{2}\left(\lambda_2\lambda_4+\lambda_3\left(\lambda_5-\bar{b}\right)\right)\sin\gamma-\frac{1}{2}\left(\lambda_3\lambda_4-\lambda_2\left(\lambda_5-\bar{b}\right)\right)\cos\gamma\\ &\quad-\frac{1}{2}\lambda_4\bar{b}\theta_2-\frac{1}{k_2-k_4}\left\{k_1 s+\left(k_3+1-\frac{1}{2}\left(\lambda_4{}^2+\lambda_5{}^2\right)\right)\theta_2\right\}\\ &\quad+\frac{1}{k_2-k_4}\left\{\frac{1}{2}\left(\lambda_3\lambda_4-\lambda_2\lambda_5\right)\sin\gamma+\frac{1}{2}\left(\lambda_2\lambda_4+\lambda_3\lambda_5\right)\cos\gamma\right\}\\ \Xi_{2\theta}&=-\frac{1}{2}\left(\lambda_2\lambda_4+\lambda_3\lambda_5\right)\sin\gamma+\frac{1}{2}\left(\lambda_3\lambda_4-\lambda_2\lambda_5\right)\cos\gamma\\ &\quad+\frac{1}{k_2-k_4}\left\{k_1 s+\left(k_3+1-\frac{1}{2}\left(\lambda_4{}^2+\lambda_5{}^2\right)\right)\theta_2\right\}\\ &\quad-\frac{1}{k_2-k_4}\left\{\frac{1}{2}\left(\lambda_3\lambda_4-\lambda_2\lambda_5\right)\sin\gamma+\frac{1}{2}\left(\lambda_2\lambda_4+\lambda_3\lambda_5\right)\cos\gamma\right\}\end{aligned} \tag{6.124}$$

Finally we obtain the equations of the second order approximation (only equations for s and θ are interesting now):

$$\begin{aligned}s_2'&=\varepsilon\left(1-\lambda_1\right)x_2+\varepsilon^2\Xi_{2s}\\ \theta_2'&=\varepsilon\lambda_1 x_2+\varepsilon^2\Xi_{2\theta}\end{aligned} \tag{6.125}$$

If we are interested in the stationary solution we obtain the following relationship

$$x_{2\infty} = -\frac{\varepsilon}{\lambda_1}\Xi_{2\theta} \tag{6.126}$$

This result can be inserted together with $\theta_{1\infty}$ into the following equation

$$\left(1-\lambda_1\right)\Xi_{2\theta}\left(s_{2\infty},\theta_{1\infty}\right) = \lambda_1\Xi_{2s}\left(s_{2\infty},\theta_{1\infty}\right) \tag{6.127}$$

It's a linear equation with respect to $s_{2\infty}$ and its solution is quite simple

$$s_{2\infty} = -\frac{k_3}{k_1}\theta_{1\infty} \tag{6.128}$$

Equations (6.115) and (6.128) give us the required stationary values for the pendulum's tilt and the cart's displacement.

6.6.7 Discussion of the Results

Let us discuss the obtained results and compare them with the numeric simulations (see Fig. 6.30).

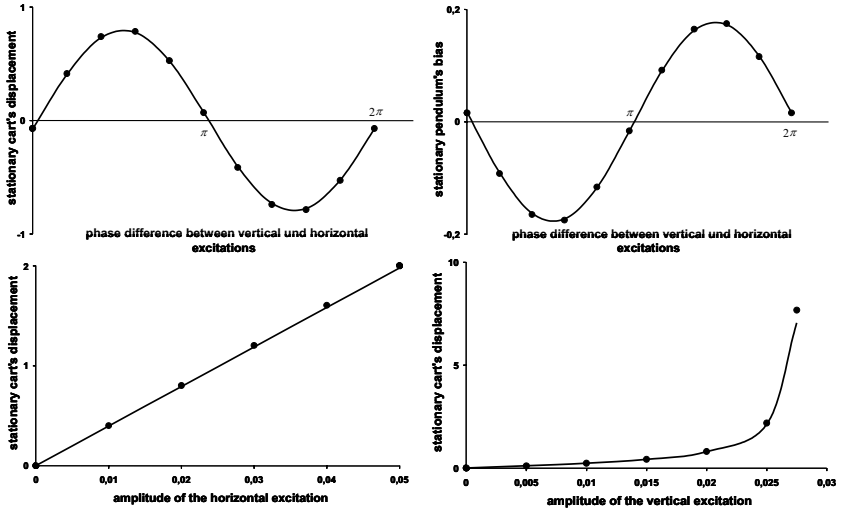

Fig. 6.30. System's performance as a function of the excitation's parameters; solid lines - analytic predictions; dots – numeric simulation of the linearized system

Fig. 6.30 demonstrates the excellent accuracy of the analytic prediction.

As appears from (6.115), a nonzero value for $\theta_{1\infty}$ and $s_{2\infty}$ can occur only with bi-directional HF excitation, i.e. when $\overline{ab} \neq 0$. This result is already known from the analysis of a simple pendulum with the HF-excited suspension point (compare the stationary pendulum bias with the relationship (6.30)).

Equation (6.128) reveals that when $k_3/k_1 > 0$ then $s_{2\infty}/\theta_{1\infty} < 0$, that is: with the controller adding positive stiffness to both the cart and the pendulum part of the system, the cart and the pendulum will be biased in opposite directions. This is a consequence of the functional (6.98) which has to be minimized by the optimal controller, including the choice of weighting constants $\mathbf{Q}$ and R. It has nothing to do with the HF excitation, but merely reflects that at equilibrium $\dot{s} = \dot{\theta} = 0$, the functional (6.98) will be minimal for $u^2(z) = 0 \Rightarrow (k_1 s + k_3 \theta)^2 = 0$, which gives the relationship (6.128).

The origin of the bias in the pendulum's motions θ, is the asymmetric term $\tilde{q}\cos(\tau + \gamma)$, which on certain conditions is nonzero even when the pendulum is vertical. This term generally does not vanish in the presence of bi-directional HF-excitation, $\overline{ab} \neq 0$, and therefore $\theta_{1\infty} = 0$ is not an equilibrium.

It is further evident from (6.115) that the primary source of the bias lies in the part of the controller that is intended to damp out oscillations about the equilibrium for the cart at $s = 0$. If $k_2 = 0$, then the right-hand side of (6.115) vanishes, and $\theta_{1\infty} = 0$ becomes a stationary quasi-equilibrium.

Another important point is the dependency of the stationary bias from the phase difference between the vertical and the horizontal excitation's components. This fact allows changing the system's equilibrium without any additional energy only by changing the phase difference. At a certain phase difference the up-pointing pendulum's position becomes the stable equilibrium as if the system would not be excited at all.

$$\gamma_0 = \arctan(k_2 - k_4) \tag{6.129}$$

The control error in the presence of HF excitation can be explained by the asymmetric forces induced by the elliptic HF-excitation. The vertical components of these forces are transmitted to the system directly by the platform, while the horizontal components are transferred primarily by the damping force $k_2\dot{s}$ determined by the controller. The only other way how the horizontal components contributing to the asymmetry can be transferred to the system is by the stiffness term $k_1 s$; however, at high frequencies the damping term is much stronger than the stiffness term, i.e. the influence of the terms depending on the velocities is dominant.

6.6.8 A Robust Control with Averaging Observer

This cognition enables analysis to find out how the proper behaviour of the system can be restored in presence of the HF-excitation. The simplest way is to prevent HF-excitation from coming into control system by means of a "slow" observer. Under the slow observer we would understand some kind of a transformer between the real values of some state variables and the input values for the control system. The simplest example for the slow observer (but surely not the only one) is the "averaging" observer or the low pass filter if we use the signal processing language. In our case we have to filter the velocity signals only. It can be done by an elementary second order filter used at the entrance of the control algorithm. Instead of the real cart's and pendulum's velocities we use the filtered ones. Instead of the original control (6.101) a new one can be applied:

$$
\begin{aligned}
&u = -k_1 s - k_2 v_s - k_3 \theta - k_1 v_\theta \\
&T_{1s}\dot{v}_s + v_s = k_{1s}\hat{v}_s \\
&T_{2s}\dot{\hat{v}}_s + \hat{v}_s = k_{2s}\dot{s} \\
&T_{1\theta}\dot{v}_\theta + v_\theta = k_{1\theta}\hat{v}_\theta \\
&T_{2\theta}\dot{\hat{v}}_\theta + \hat{v}_\theta = k_{2\theta}\dot{\theta}
\end{aligned}
\tag{6.130}
$$

This filter is surely not optimal one, neither is the corresponding control. But it is good enough in order to demonstrate the efficiency of the averaging observer (see Fig. 6.31)

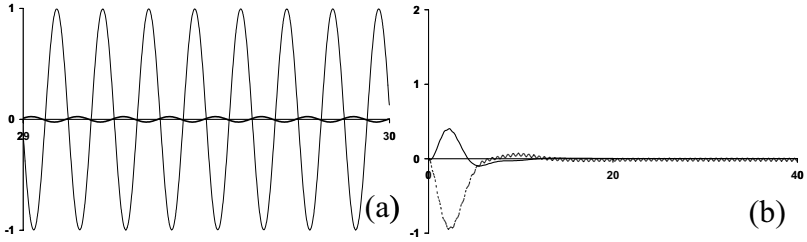

Fig. 6.31. Simulation results for the control based on the filtered estimates for the velocities; (a) – the efficiency of filtering; thin line – the cart's velocity; thick line – the filtered cart's velocity; (b) – the performance of the control system – both the cart and the pendulum are rapidly stabilized in the right positions

It would not be difficult to design the corresponding optimal filter and optimal control applying the same methods we have used for the initial control design, but it is not the objective of our analysis. The only disadvantage of the averaging observer is obvious. The control system looses a lot of information about the system's state, which cannot be used if there are some additional functions. The only sensible approach is to distinguish between observations used to ensure the stable

and robust behaviour of the controlled object and perhaps the same observations used for other purposes.

7. Systems with High-Frequency Excitation: Advanced Analysis and Generalizations

The basic effects of the strong HF excitation were discussed in the previous chapter. The exceptional role of the large oscillating terms was also mentioned in paragraph 6.4 but this case was excluded from our analysis. The technical reason for this restriction is quite simple. Particular systems with strong excitation commonly occur in various applications. Usually these systems are used if we are interested in analyzing motions of a machine where the inertia of its housing is significantly larger than the inertia of its moving parts.

If however the mass or inertia of the mechanism's moving parts is not small (as in the case for many modern machines like cranks mechanisms or vane pumps), equations of motion containing fast oscillating inertia coefficients will be obtained. These equations contain large, fast oscillating terms depending not only on the generalized coordinates but also on the system's generalized velocities.

Another example of systems with strong excitation dependent upon the first derivative of the unknown function appears if we investigate vibrations or wave propagation in inhomogeneous media. For example, the longitudinal waves in a rod with a periodic or quasi-periodic structure. In this case the typical equations with strong excitation of general form appear naturally with respect to the spatial coordinates. Equations with the slowly modulated HF excitation are typical for this group of applications.

The general asymptotic procedure will be established in this chapter. It is suitable for analyzing strongly excited systems and those in which the periodic HF excitation depends on velocity. This analysis is illustrated by two mathematical examples highlighting the principal differences between the effect of the positional HF excitation and the effect of the terms depending on velocity. The response of an elementary nonlinear system subjected to strong external and parametric (depending on the velocity) excitation, is used to demonstrate the effects in a mechanical application. The same example is used to demonstrate how the performed analysis can be generalized for systems with very strong external excitation. Dynamics of a double pendulum with a fast rotating second link is considered as an example demonstrating some technical applications of the discussed methods.

7.1 Systems with Strong Excitation. General Analysis

The general mathematical approach for asymptotic analysis of mechanical systems with strong general HF excitation can be formulated using the following theorem (the strict mathematical formulation and the corresponding proof is given in Appendix VIII):

Consider a system

$$\ddot{x} = F(x,\dot{x},t,\tau) + \omega\Phi(x,\dot{x},t,\tau)$$
$$x\big|_{t=0} = x_0; \dot{x}\big|_{t=0} = v_0; \tau = \omega t. \tag{7.1}$$

Suppose:

1. x *is a* n*-dimensional vector of the system's generalized coordinates.*
2. *Function* F *is continuous together with its first partial derivatives with respect to all arguments.*
3. *Function* Φ *is continuous together with its second partial derivatives with respect to all arguments.*
4. *All the functions are* 2π*-periodic with respect to* τ*,* $\langle\ \rangle$ *means averaging with respect to* τ*,* $\omega >> 1$ *is the large parameter.*
5. *The general* 2π*- periodic with respect to* τ *bounded solution to the system of* n *first order differential equations is known:*

$$\frac{\partial u}{\partial \tau} = \Phi(X,u,t,\tau), \langle u(X,t,\tau)\rangle = \dot{X} \Rightarrow u = U\left(X,\dot{X},t,\tau\right) \tag{7.2}$$

It satisfies the Lipschitz conditions together with its first partial derivatives with respect to X *and* $\dot{X}$*.*

Consider a system of ordinary differential equation, which does not contain the fast time τ *alongside* (7.1):

$$M\left(X,\dot{X},t\right)\ddot{X} = V\left(X,\dot{X},t\right);$$
$$X\big|_{t=0} = x_0, \dot{X}\big|_{t=0} = v_0 - \partial\Psi/\partial\tau\big|_{t=0,\tau=0} \tag{7.3}$$

Here

$$M\left(X,\dot{X},t\right)=\left\langle W_*^T\frac{\partial U}{\partial \dot{X}}\right\rangle;\qquad \Psi=\int_0^\tau\left(U-\dot{X}\right)d\tau$$
$$V\left(X,\dot{X},t\right)=\left\langle W_*^T F\left(X,U,t,\tau\right)\right\rangle \tag{7.4}$$
$$+\left\langle W_*^T\left\{\frac{\partial\Phi}{\partial X}\Psi-\frac{\partial U}{\partial X}\dot{X}-\frac{\partial U}{\partial t}\right\}\right\rangle$$

W_* is the fundamental matrix of solutions for the linear homogeneous system

$$\frac{\partial W_*}{\partial\tau}=-\left(\frac{\partial\Phi}{\partial\dot{x}}\right)^T W_*;\quad \det\left\langle W_*^T\right\rangle\neq 0 \tag{7.5}$$

Then solutions to (7.1) *and* (7.3) *are asymptotically close to each other for the time interval* $t=O(1)\Leftrightarrow\tau=O(\omega)$.

We are going to demonstrate the origins of this result by applying formally the multiple scales technique. The analysis based on the generalized averaging leads to the mathematical proof of this theorem which can be found in Appendix VIII.

Let us start with the multiple scales and convert from the system of ordinary differential equations (7.1) to a system with partial derivatives and two independent variables t and τ:

$$\frac{\partial^2\varphi}{\partial t^2}+2\omega\frac{\partial^2\varphi}{\partial t\partial\tau}+\omega^2\frac{\partial^2\varphi}{\partial\tau^2}=F\left(\varphi,\frac{\partial\varphi}{\partial t}+\omega\frac{\partial\varphi}{\partial\tau},t,\tau\right)$$
$$+\omega\Phi\left(\varphi,\frac{\partial\varphi}{\partial t}+\omega\frac{\partial\varphi}{\partial\tau},t,\tau\right) \tag{7.6}$$

The relationship between (7.1) and (7.6) is given by the following condition: If $\varphi(t,\tau)$ is a solution to the equation (7.6), then this solution taken along the straight line $\tau=\omega t$, i.e. $x=\varphi(t,\omega t)$ is a solution to the equation (7.1). In other words, system (7.6) is more general than the equation (7.1). It gives us some freedom to choose the boundary conditions for this system. The only restriction is that the straight line $\tau=\omega t$ should be in the inner part of the considered area. We require $\varphi(t,\tau)$ being a 2π - periodic function of τ and try to find $\varphi(t,\tau)$ as a formal asymptotic expansion in terms of the small parameter $\varepsilon=1/\omega$:

$$\varphi(t,\tau)=\psi_0(t,\tau)+\varepsilon\psi_1(t,\tau)+\varepsilon^2\psi_2(t,\tau)+\ldots \tag{7.7}$$

Substituting this expansion into (7.6) and balancing the terms with equal orders of ε we obtain:

$$\varepsilon^{-2}: \frac{\partial^2 \psi_0}{\partial \tau^2} = 0, \tag{7.8}$$

$$\varepsilon^{-1}: \frac{\partial^2 \psi_1}{\partial \tau^2} + 2\frac{\partial^2 \psi_0}{\partial t \partial \tau} = \Phi\left(\psi_0, \frac{\partial \psi_1}{\partial \tau} + \frac{\partial \psi_0}{\partial t} + \omega \frac{\partial \psi_0}{\partial \tau}, t, \tau\right), \tag{7.9}$$

$$\varepsilon^{0}: \frac{\partial^2 \psi_2}{\partial \tau^2} + 2\frac{\partial^2 \psi_1}{\partial t \partial \tau} + \frac{\partial^2 \psi_0}{\partial t^2} = F + \frac{\partial \Phi}{\partial x}\psi_1 + \frac{\partial \Phi}{\partial \dot{x}}\left(\frac{\partial \psi_1}{\partial t} + \frac{\partial \psi_2}{\partial \tau}\right) \tag{7.10}$$

The last step must be justified, because the second argument of all the functions on the right hand side of the equations is:

$$\frac{\partial \psi_1}{\partial \tau} + \frac{\partial \psi_0}{\partial t} + \omega \frac{\partial \psi_0}{\partial \tau} \tag{7.11}$$

This expression can take values of the magnitude order of the large parameter ω. So it can create terms of any order in our equations depending on how Φ depends on $\dot{x}$. If we require Φ, as usual, being a bounded function in the vicinity of the solution to the averaged system, we reduce the problem to the *a posteriori* check of our assumptions concerning the magnitude order of $\dot{x}$ in the vicinity of the found solution.

In the considered case of strong excitation the problem is insignificant. The general solution to the equation (7.8) has the following form:

$$\psi_0(t, \tau) = X(t) + A(t)\tau \tag{7.12}$$

According to the periodicity of ψ_0, we get $A(t) = 0$. Hence, $\psi_0 = X(t)$. It depends only on the slow time t and the large terms in the equations (7.9) and (7.10) disappear automatically. However, it is just this problem that prevents a general analysis of systems with very strong excitation.

The objective of the following analysis is to obtain differential equations governing the still unknown function $X(t)$. These equations should not contain the fast time τ.

The equation (7.9) after substituting in it the solution of the equation (7.8) takes the form:

$$\frac{\partial^2 \psi_1}{\partial \tau^2} = \Phi\left(X, \dot{X} + \frac{\partial \psi_1}{\partial \tau}, t, \tau\right) \tag{7.13}$$

It is natural to call this equation "*The Equation of Fast Motion*". It is a differential equation with only partial derivatives with respect to τ. So we can take $X, \dot{X}$ and t to be constant parameters while solving (7.13). Referring to the fifth condition of the Theorem (*cf.* (7.2)) we can rewrite the solution of the equation (7.13) as follows:

$$\psi_1 = \Psi(t,\tau) + X_1(t);\ \Psi = \int_0^{\tau} \left(U - \dot{X}\right)\, d\tau \tag{7.14}$$

The new unknown function $X_1(t)$ is a small slow correction to the main slow part of the solution $X(t)$. The system (7.2) is significantly simpler than the original system (7.1), because its order is halved and we can take all the functions of slow time t to be constant. In other words we are not interested in the slow evolution of the system here but only in its high frequency oscillations.

Let us move on now to the equation (7.10) which can be now rewritten as follows:

$$\frac{\partial v}{\partial \tau} = \frac{\partial \Phi}{\partial \dot{x}} v + F + \frac{\partial \Phi}{\partial x}\psi_1 - \frac{\partial^2 \psi_1}{\partial t \partial \tau} - \ddot{X} \tag{7.15}$$

The new unknown function was introduced here:

$$v = \frac{\partial \psi_2}{\partial \tau} + \frac{\partial \psi_1}{\partial t} \tag{7.16}$$

This is a system of n first order linear inhomogeneous equations with periodic coefficients. As it is known from the general theory of linear systems with periodic coefficients, the projections of the inhomogeneous parts of the equations on the solutions of the system conjugated to the homogeneous part of the original one must vanish in order the periodic solutions to (7.15) exist. (A reference to the classical work of Poincarè [98] seems to be in order here and to underscore the close relationship of Poincarè's method to that of multiple scales, which is in this case a procedure to find a periodic solution with respect to τ (7.6) with a non-isolated unperturbed solution (7.12).

This necessary condition takes the following form:

$$\left\langle W_*^T \left\{ F + \frac{\partial \Phi}{\partial x}\psi_1 - \frac{\partial^2 \psi_1}{\partial t \partial \tau} - \ddot{X} \right\} \right\rangle = 0 \tag{7.17}$$

The matrix W_* is defined as the fundamental matrix of the homogeneous system (7.5). The equation (7.17) contains seemingly two unknown functions X and

X_1. This statement becomes clear if we replace ψ_1 by its explicit form (7.14). Then (7.17) can be rewritten as follows:

$$\left\langle W_*^T \left\{ F + \frac{\partial \Phi}{\partial x}\Psi + \frac{\partial \Phi}{\partial x}X_1 - \frac{\partial^2 \Psi}{\partial t \partial \tau} - \ddot{X} \right\} \right\rangle = 0 \tag{7.18}$$

Let us show that the term which depends on the X_1 is equal to zero. In other words we are going to prove the following identity:

$$\left\langle W_*^T \frac{\partial \Phi}{\partial x} \right\rangle = 0 \tag{7.19}$$

Let us use the equation (7.2) written as an identity:

$$\frac{\partial u}{\partial \tau} = \Phi\left(X, u\left(X, \dot{X}, t, \tau\right), t, \tau\right) \tag{7.20}$$

Deriving it with respect to X one obtains:

$$\frac{\partial^2 u}{\partial X \partial \tau} = \frac{\partial \Phi}{\partial X} + \frac{\partial \Phi}{\partial \dot{X}}\frac{\partial u}{\partial X} \tag{7.21}$$

Hence we can rewrite (7.19) as follows:

$$\left\langle W_*^T \frac{\partial \Phi}{\partial X} \right\rangle = \left\langle W_*^T \frac{\partial^2 u}{\partial X \partial \tau} - W_*^T \frac{\partial \Phi}{\partial \dot{X}}\frac{\partial u}{\partial X} \right\rangle \tag{7.22}$$

Due to the periodicity of W_* and u we can write:

$$\begin{aligned} \left\langle W_*^T \frac{\partial^2 u}{\partial X \partial \tau} \right\rangle &= \frac{1}{2\pi}\int_0^{2\pi} W_*^T \frac{\partial^2 u}{\partial X \partial \tau} d\tau \\ &= \frac{1}{2\pi}\left(W_*^T \frac{\partial u}{\partial X} \right)\Bigg|_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial W_*^T}{\partial \tau}\frac{\partial u}{\partial X} d\tau \\ &= \frac{1}{2\pi}\int_0^{2\pi} W_*^T \frac{\partial \Phi}{\partial \dot{X}}\frac{\partial u}{\partial X} d\tau = \left\langle W_*^T \frac{\partial \Phi}{\partial \dot{X}}\frac{\partial u}{\partial X} \right\rangle \end{aligned} \tag{7.23}$$

Substituting (7.23) into (7.22) we obtain the required identity (7.19).
Due to this result we can simplify the equation (7.17) as follows:

$$\left\langle W_*^T \right\rangle \ddot{X} = \left\langle W_*^T \left\{ F + \frac{\partial \Phi}{\partial x} \Psi - \frac{\partial^2 \Psi}{\partial t \partial \tau} \right\} \right\rangle \tag{7.24}$$

Lastly, function Ψ depends on t both directly and indirectly through functions $X(t)$ and $\dot{X}(t)$. Under $\partial/\partial t$ we understand here the "full" partial derivative with respect to t. Taking this into account and using the "partial" partial derivatives we obtain the final explicit form of the equation (7.24):

$$\begin{gathered} \frac{\partial \Psi}{\partial \tau} = U + \dot{X} \Rightarrow \frac{\partial^2 \Psi}{\partial t \partial \tau} = \frac{\partial U}{\partial t} + \frac{\partial U}{\partial X} \dot{X} + \frac{\partial U}{\partial \dot{X}} \ddot{X} + \ddot{X} \Rightarrow \\ \left\langle W_*^T \frac{\partial U}{\partial \dot{X}} \right\rangle \ddot{X} = \left\langle W_*^T \left\{ F + \frac{\partial \Phi}{\partial x} \Psi - \frac{\partial U}{\partial t} - \frac{\partial U}{\partial X} \dot{X} \right\} \right\rangle \end{gathered} \tag{7.25}$$

This is exactly the equation (7.4) required in the theorem. Equation (7.4) does not contain the fast time, and determines the slow evolution of the solutions to the original system (7.1). That's why they could be called "*The Equations of Slow Motion*". The Function $V(X, \dot{X}, t)$ can be naturally called the vibrational force and the matrix $M(X, \dot{X}, t)$ can be interpreted as a matrix of the system's "*effective mass with respect to slow motions*". This matrix depends on the solution of the equations of fast motion, i.e. on the amplitude of the fast excitation.

If the function Φ does not depend on $\dot{x}$, system (7.3),(7.4) goes over into equations, well known, for example, from the works of Blekhman [20]:

$$\ddot{X} = \left\langle F + \frac{\partial \Phi}{\partial x} \Psi \right\rangle \tag{7.26}$$

This is nothing different but the equation (6.58) which was already obtained in the section 6.4 with the help of the standard averaging.

An interesting question is, can the general result (7.4) be obtained by the standard averaging or by some modified averaging? It was already shown in the previous chapter that the standard approach based on the variation of the integrals of the unperturbed system, does not help to obtain the general relationships similar to (7.26). We know however, that the solution we are interested in can be represented at the coordinates' level, as a superposition of the small fast oscillations and large slow evolutional motions. At the velocities' level the fast oscillations and the slow evolution are of the same order of magnitude.

The generalized averaging procedure based on this fact can be found in Appendix VIII. It leads directly to the mathematical proof of the formal equations (7.3), (7.4).

Let us give two mathematical examples illustrating the effect of the fast oscillating terms depending on the velocities.

7.2 Two Mathematical Examples of Systems with Strong Excitation

Now we are going to consider two mathematical examples of systems with strong excitation, illustrating the main qualitative difference between the systems with strong terms depending on the velocities (7.1) and systems in which these terms depend only on the co-ordinates.

7.2.1 A System with One Degree of Freedom and Strong HF Excitation Depending on the Velocity

Firstly, following [34] we consider a system with one degree of freedom. Secondly an example of a system with two degrees of freedom is analyzed. It is necessary to emphasize that these examples are purely mathematical. Further mechanical examples are discussed in paragraph 7.3.

As in the first example we are going to analyze the following equation:

$$\ddot{x} + \beta \dot{x} + x = a\omega \dot{x} \cos(\omega t) \tag{7.27}$$

Let us assume that a and β are both of the magnitude order 1 and $\omega >> 1$, is the large parameter. Then according to our designations

$$F = -\beta \dot{x} - x; \quad \Phi = a\dot{x} \cos \tau \tag{7.28}$$

The equation of fast motion takes the form:

$$\frac{\partial u}{\partial \tau} = a\, u \cos \tau \tag{7.29}$$

Its periodic solution satisfying the average condition (7.2) is:

$$u = \frac{\dot{X}}{I_0(a)} e^{a \sin \tau} \tag{7.30}$$

Here $I_0(a)$ is the modified Bessel's function of 0 -order:

$$I_0(a) = \frac{1}{2\pi} \int_0^{2\pi} e^{a \sin \vartheta} d\vartheta \tag{7.31}$$

The linear homogeneous system (4.2.6) can be written down as follows:

$$\frac{\partial W_*}{\partial \tau} = -a\, W_* \cos \tau \tag{7.32}$$

Its periodic solution is:

$$W_* = e^{-a\sin\tau}, \quad W_* > 0 \tag{7.33}$$

Substituting these expressions into (7.4) and averaging the corresponding terms, one obtains the following equation of slow motion:

$$\ddot{X} + \beta \dot{X} + I_0^{\,2}(a)X = 0 \tag{7.34}$$

Equation (7.34) is a very simple linear differential equation with constant coefficients. Its solution, together with the fast component (7.30) gives us the approximate solution to (7.27).

Let us demonstrate how the same result can be obtained by the standard averaging based on the variation of the integrals of the unperturbed equations. Converting to the fast time as the independent variable one can rewrite (7.27) as follows:

$$\begin{aligned} x' &= \varepsilon v \\ v' &= -\varepsilon\beta v - \varepsilon x + av\cos\tau \end{aligned} \tag{7.35}$$

The corresponding unperturbed system is

$$\begin{aligned} x_0' &= 0 \\ v_0' &= av_0\cos\tau \end{aligned} \tag{7.36}$$

Its general solution is periodic:

$$x_0 = const;\ v_0 = z_0 e^{a\sin\tau};\ z_0 = const \tag{7.37}$$

We can consider (7.37) as the variable transformation:

$$v = ze^{a\sin\tau} \tag{7.38}$$

The following equations govern the new variables:

$$\begin{aligned} x' &= \varepsilon z e^{a\sin\tau} \\ z' &= -\varepsilon\beta z - \varepsilon x e^{-a\sin\tau} \end{aligned} \tag{7.39}$$

This is a system in the standard form. It can be averaged directly. The result is:

$$\begin{aligned} x_1' &= \varepsilon z_1 I_0(a) \\ z_1' &= -\varepsilon\beta z_1 - \varepsilon x_1 I_0(a) \end{aligned} \tag{7.40}$$

Converting to a second order differential equation and returning back to the slow time as an independent variable one obtains the equation (7.34).

A comparison of the analytic solution with the numeric one can be found in Fig. 7.1 and Fig. 7.2 for two different values of the large parameter (ω = 50 in

Fig.7.1; ω = 10 in Fig.7.2). In both cases the calculations were performed for the following parameters' values: $\beta = 0, \quad a = 1 \Rightarrow I_0(1) \approx 1,266$

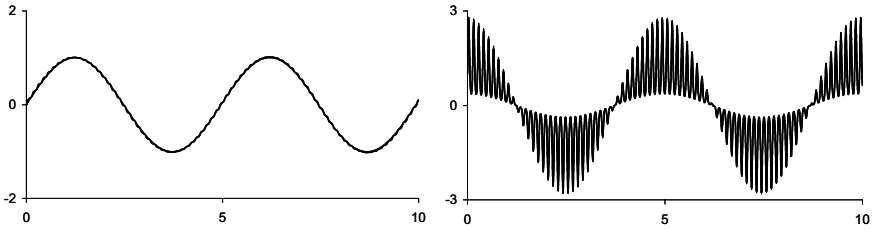

Fig. 7.1. Comparison between analytic (thin line) and numeric (thick line) solutions

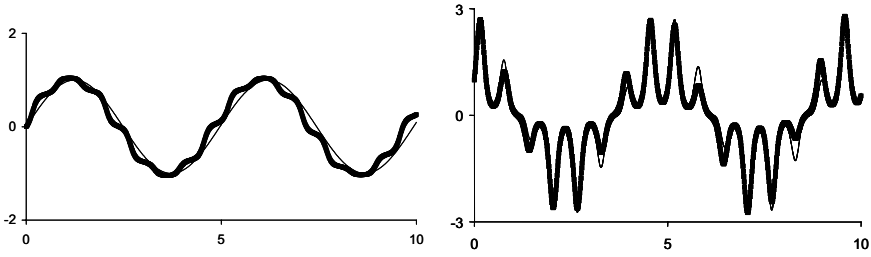

Fig. 7.2. Comparison between analytic (thin line) and numeric (thick line) solutions for the smaller values of the large parameter

In the first case there is no visible difference between two predictions. For smaller values of ω the trajectories diverge slightly.

Consider some features of this solution.

- It is typical for systems with strong excitation that the solution is a superposition of slow component and fast oscillations. These are small respectively to the generalized coordinates x, but their derivatives are not small respectively to the generalized velocities $\dot{x}$.
- All the particular results which can be achieved according to the general approach (7.3), (7.4) can be also obtained by multiple scales or standard averaging. These general purpose methods lead however only to particular results. The general relationships clarify the main difference between systems with the HF-excitation depending on the velocities and those with the HF-excitation depending only on the positional coordinates. This difference can be formulated very simply:

> The HF-excitation depending on the velocities changes the effective mass of the system with respect to slow motions.

- The large fast oscillations of the damping coefficient change the frequency of the slow free oscillations of the system. This frequency can be controlled by the

excitation's amplitude. This result is similar to that reported by Nayfeh and Nayfeh [81] for another particular system. This effect is typical for systems with strong excitation, but in this particular case it is caused by the changed effective mass of the system. This thesis is even clearer illustrated by the next example.

7.2.2 A System with Two Degrees of Freedom and a Skew Symmetric HF Excitation Depending on the Velocities

We are going to consider a system with two degrees of freedom as the second mathematical example.

$$\begin{aligned} \ddot{x}_1 &= a\omega\dot{x}_2 \cos\omega t + F_1\left(x_1, x_2\right) \\ \ddot{x}_2 &= -a\omega\dot{x}_1 \cos\omega t + F_2\left(x_1, x_2\right) \end{aligned} \tag{7.41}$$

The characteristic for this system is the skew-symmetric matrix of the HF excitation. Let us assume that a has the magnitude order 1; $\omega >> 1$ is the large parameter. F_1 and F_2 are arbitrary bounded functions describing slow forces. The following relationships are valid according to our notation:

$$\mathbf{\Phi} = \mathbf{Q}a\dot{x}\cos\tau \quad ; F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}; \quad \mathbf{Q} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{7.42}$$

The equation of fast motion takes the form:

$$\frac{\partial u}{\partial \tau} = \mathbf{Q}au\cos\tau \tag{7.43}$$

Its periodic solution satisfying the average condition (7.2) is:

$$U = \frac{1}{J_0\left(a\right)}\mathbf{W}_*\left(\tau\right)\dot{X} \tag{7.44}$$

Here $J_0\left(a\right)$ is the Bessel's function of 0 -order:

$$J_0\left(a\right) = \frac{1}{2\pi}\int_0^{2\pi}\cos\left(a\sin\tau\right)d\tau \tag{7.45}$$

$\mathbf{W}_*$ is the fundamental matrix to the linear homogeneous system (7.5) taking in this particular case, the form

$$\frac{\partial \mathbf{W}_*}{\partial \tau} = -\mathbf{Q}\mathbf{W}_* a\cos\tau \tag{7.46}$$

This equation can be easily solved. Let us write it down in components:

$$\begin{aligned} &\frac{\partial W_*^{11}}{\partial \tau} = -W_*^{21} a\cos\tau;\ \frac{\partial W_*^{12}}{\partial \tau} = -W_*^{22} a\cos\tau \\ &\frac{\partial W_*^{21}}{\partial \tau} = W_*^{11} a\cos\tau;\ \frac{\partial W_*^{22}}{\partial \tau} = W_*^{12} a\cos\tau \end{aligned} \tag{7.47}$$

This system can be transformed as follows:

$$\begin{aligned} &\frac{\partial W_*^{11}}{\partial (a\sin\tau)} = -W_*^{21};\ \frac{\partial W_*^{21}}{\partial (a\sin\tau)} = W_*^{11} \\ &\frac{\partial W_*^{12}}{\partial (a\sin\tau)} = -W_*^{22};\ \frac{\partial W_*^{22}}{\partial (a\sin\tau)} = W_*^{12} \end{aligned} \tag{7.48}$$

The fundamental solution to each pair of the equations is obvious:

$$\mathbf{W}_* = \begin{bmatrix} \cos(a\sin\tau) & \sin(a\sin\tau) \\ -\sin(a\sin\tau) & \cos(a\sin\tau) \end{bmatrix} \tag{7.49}$$

Substituting these expressions into (7.4) we average the corresponding terms in order to obtain the equations governing the slow motion:

$$\begin{aligned} \mathbf{M} &= \left\langle \mathbf{W}_*^T \frac{\partial U}{\partial \dot{X}} \right\rangle = \frac{1}{J_0(a)} \left\langle \mathbf{W}_*^T \mathbf{W}_* \right\rangle = \frac{1}{J_0(a)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ V &= \left\langle \mathbf{W}_*^T \right\rangle F = \begin{bmatrix} \left\langle \cos(a\sin\tau) \right\rangle & \left\langle \sin(a\sin\tau) \right\rangle \\ -\left\langle \sin(a\sin\tau) \right\rangle & \left\langle \cos(a\sin\tau) \right\rangle \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \\ &= J_0(a) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \end{aligned} \tag{7.50}$$

The equation of the slow motion is:

$$\ddot{X} = J_0^2(a) F(X) \tag{7.51}$$

This system is extremely interesting. It is common for systems with fast excitation, which does not depend on the generalized velocities that some additional

slow vibration forces expressing the averaged result of the HF excitation appear in the equations of the slow motion. These terms are usually added to the existing slow forces. Here we have got a totally different result. The existing slow forces are multiplied by a factor, which depends on the amplitude of fast excitation. Further, due to the oscillating character of the Bessel's function J_0, it has an infinite number of zeros. This means that there is a countable number of excitation amplitudes corresponding to these zeros, for which the equation of the slow motion takes the form $\ddot{X} = 0$ for arbitrary functions F, i.e. the system does not react to an arbitrary slow action. This unusual property is in some sense similar to the quantization effect in micro-physics. It is interesting to notice that this effect still remains for the higher order approximations.

The orbit calculated numerically for the first zero of the Bessel's function is shown in Fig. 7.3.

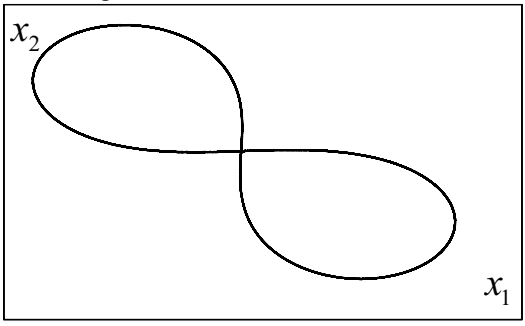

Fig. 7.3. Stationary orbit for a = 2.40482 and ω =300.

It must be emphasized that the system (7.41) does not exist in mechanics and the considered example has only a pure mathematical character. Nevertheless it demonstrates clearly the principal difference between systems containing fast excitation terms depending on the velocities and those, which do not contain these terms. The obtained result is quite unexpected and it would be interesting to analyze if there are physical systems described by these or similar equations.

Let us now move on to some physically sensible examples illustrating the influence of the strong HF excitation depending on the velocities.

7.3 The Lowest Natural Frequencies of an Elastic Rod with Periodic Structure

Consider longitudinal waves in a rod with periodic structure and fixed ends. It can be described by a differential equation with partial derivatives and corresponding boundary conditions:

$$\rho(\lambda x)\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left(E(\lambda x)A(\lambda x)\frac{\partial u}{\partial x}\right)=0$$
$$u\big|_{x=0}=u\big|_{x=l}=0 \tag{7.52}$$

The density of the rod, its elastic modulus and the cross section area presumed to be some fast oscillating periodic functions of the spatial coordinate x :

$$\begin{aligned}&\rho\left(\lambda\left(x+2\pi\right)\right)=\rho(\lambda x)\\&E\left(\lambda\left(x+2\pi\right)\right)=E(\lambda x)\\&A\left(\lambda\left(x+2\pi\right)\right)=A(\lambda x)\\&\lambda>>\frac{1}{l}\end{aligned} \tag{7.53}$$

The same equations are valid for any linear problem of wave propagation in a one dimensional periodic continuum. We are now going to find the approximate expressions for the lowest natural frequencies of the rod. Applying the standard procedure we try to find a solution to (7.52) in the following form:

$$u(x,t)=v(x)\sin\omega t \tag{7.54}$$

Inserting (7.54) into (7.52) and balancing the corresponding terms the following ordinary differential equation can be obtained:

$$\frac{d^2v}{dx^2}+\omega^2\frac{\rho(\lambda x)}{E(\lambda x)A(\lambda x)}v=-\lambda\frac{d\ln\left(E(\lambda x)A(\lambda x)\right)}{d(\lambda x)}\frac{dv}{dx}$$
$$v(0)=v(l)=0 \tag{7.55}$$

This is an equation with strong excitation (λ is the large parameter) considered in the section 7.1 with respect to the spatial coordinate x. Referring back to the general form (7.1) we can use the following notation:

$$F=-\omega^2\frac{\rho(\xi)}{E(\xi)A(\xi)}v;\quad \Phi=-\frac{d\ln\left(E(\xi)A(\xi)\right)}{d\xi}\frac{dv}{dx};\quad \xi=\lambda x \tag{7.56}$$

The corresponding equation of the fast motion has the form:

$$\frac{\partial p}{\partial\xi}=-\frac{d\ln\left(E(\xi)A(\xi)\right)}{d\xi}p \tag{7.57}$$

Its solution fulfilling the average condition $\langle p \rangle = V'$ is

$$p = \frac{V'}{E(\xi)A(\xi)} \left\langle \frac{1}{E(\xi)A(\xi)} \right\rangle^{-1} \tag{7.58}$$

The system conjugated to the homogeneous part of the equation of the second approximation is

$$\frac{\partial W_*}{\partial \xi} = \frac{d \ln\left(E(\xi)A(\xi)\right)}{d\xi} W_* \tag{7.59}$$

Its fundamental solution is easy to find:

$$W_* = E(\xi)A(\xi) \tag{7.60}$$

Averaging the corresponding terms one obtains:

$$M = \left\langle \frac{1}{E(\xi)A(\xi)} \right\rangle^{-1}; V = -\omega^2 \langle \rho(\xi) \rangle \tag{7.61}$$

Finally, we receive the equation of "slow" motion, which means in this case the long wave approximation:

$$\begin{aligned} &V'' + \omega^2 \langle \rho(\xi) \rangle \left\langle \frac{1}{E(\xi)A(\xi)} \right\rangle V = 0 \\ &V(0) = V(l) = 0 \end{aligned} \tag{7.62}$$

The natural frequencies of the rod and the corresponding wave propagation velocity follow immediately from this equation:

$$\begin{aligned} &c = \sqrt{\frac{1}{\langle \rho(\xi) \rangle} \left\langle \frac{1}{E(\xi)A(\xi)} \right\rangle^{-1}} \\ &\omega = \frac{\pi n c}{l}; \quad n = 1, 2, \ldots \end{aligned} \tag{7.63}$$

This result can be obtained by means of different perturbation methods. We are going to consider several more complicated examples in the next sections.

7.4 Response of a One Degree of Freedom Nonlinear System to a Strong HF External and Parametric Excitation Due to Oscillating Inertia

Now we are going to consider an example, which appears naturally in systems with oscillating inertia coefficients.

7.4.1 The Governing Equations and Their Transformation to the Basic Mathematical Form

Consider the following system:

$$\frac{d}{dt}\left(\frac{\dot{x}}{1+c(t)\cos\omega t}\right)+x+\alpha x^3=\omega f(t)\sin(\omega t+\theta) \tag{7.64}$$

This equation describes a one degree of freedom system with oscillating inertia. An example of a mechanical system which in an appropriate approximation can be described by an equation similar to (7.64) is shown in Fig. 7.4. It is a pendulum with a mass moving along it.

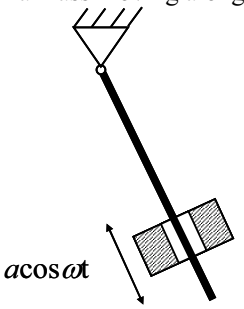

Fig. 7.4. A pendulum with the fast oscillating length is the elementary example of a system with the HF parametric excitation depending on the velocity

Its equation of motion can be written down as follows:

$$\frac{d}{dt}\left(\left(l+a\cos\omega t\right)^2\dot{\varphi}\right)+g\left(l+a\cos\omega t\right)\sin\varphi=0 \tag{7.65}$$

Here l is the unperturbed length of the pendulum. We can transform this equation to a form very similar to (7.64) if we presume that the amplitude of the length's oscillations is small in comparison with the unperturbed length and replace the sine function by its cubic approximation:

$$\frac{a}{l}<<1;\omega>>1;\ \frac{a}{l}\omega=O(1);\ \sin\varphi\approx\varphi-\frac{\varphi^3}{6}\Rightarrow$$
$$\frac{d}{dt}\left(\left(1+2\frac{a}{l}\cos\omega t\right)\dot{\varphi}\right)+\frac{g}{l}\left(\varphi-\frac{\varphi^3}{6}\right)=0 \tag{7.66}$$

The asymptotic accuracy of this equation would not change if we notice the following relationship:

$$1+2\frac{a}{l}\cos\omega t=\frac{1}{1-2\frac{a}{l}\cos\omega t}+O\left(\left(\frac{a}{l}\right)^2\right)\Rightarrow$$
$$\frac{d}{dt}\left(\frac{\dot{\varphi}}{1-2\frac{a}{l}\cos\omega t}\right)+\frac{g}{l}\left(\varphi-\frac{\varphi^3}{6}\right)=0 \tag{7.67}$$

This is exactly the form of the parametric excitation depending on the velocity that we are going to investigate in the equation (7.64). We will not assume the excitation's amplitude to be small. The form (7.64) allows highlighting the most important general properties of such a system avoiding the unnecessary mathematical complications. We also suppose, according to the previous chapter, the excitation's amplitude to be slowly modulated.

7.4.2 Obtaining the Equations Governing Slow Motion

The equation (7.64) can be rewritten as follows:

$$\ddot{x}=-(x+\alpha x^3)(1+c\cos\tau)+\frac{\dot{x}\dot{c}\cos\tau}{1+c\cos\tau}$$
$$+\omega f\sin(\tau+\theta)(1+c\cos\tau)-\frac{\omega\dot{x}c\sin\tau}{1+c\cos\tau}. \tag{7.68}$$

Referring back to the general form (7.1) we can use the notation:

$$F=-(x+\alpha\ \ x^3)(1+c\cos\tau)+\frac{\dot{x}\dot{c}\cos\tau}{1+c\cos\tau}$$
$$\Phi=f\sin(\tau+\theta)(1+c\cos\tau)-\frac{\dot{x}c\sin\tau}{1+c\cos\tau} \tag{7.69}$$

The corresponding equation of fast motion is:

$$\frac{\partial u}{\partial \tau} = -\frac{uc\sin\tau}{1+c\cos\tau} + f\sin(\tau+\theta)(1+c\cos\tau) \tag{7.70}$$

Its solution which fulfils the average condition $\langle u \rangle = \dot{X}$ is

$$u = \left(\dot{X} + \tfrac{1}{2} fc\cos\theta - f\cos\left(\tau+\theta\right)\right)\left(1+c\cos\tau\right) \tag{7.71}$$

The system conjugated to the homogeneous part of the equation of second approximation determines the weighting function:

$$\frac{\partial W_*}{\partial \tau} = \frac{cW_*\sin\tau}{\left(1+c\cos\tau\right)} \Rightarrow W_* = \frac{1}{\left(1+\cos\tau\right)}. \tag{7.72}$$

Averaging the corresponding terms one obtains:

$$M = 1;\ V = -X - \alpha X^3 - \tfrac{1}{2}\left(\dot{f}c + f\dot{c}\right)\cos\theta \tag{7.73}$$

Finally, we receive the equation governing the slow motions of the system:

$$\ddot{X} + X + \alpha X^3 = -\frac{1}{2}\frac{d\left(fc\right)}{dt}\cos\theta \tag{7.74}$$

7.4.3 Discussion of the Results

This equation has several interesting peculiarities. Firstly, in this case neither the effective mass, nor the natural frequency gets transformed. Instead, we have another interesting phenomenon – the transformation of the slow excitation's character. There are both external and parametric high frequency excitations in the original system. If the high frequency excitations are slowly modulated by the functions $f\left(t\right)$ and $c\left(t\right)$, it means that we do not only deal with the high frequency but also with the low frequency parametric and external excitations of the original system. The parametric excitation has completely disappeared in the equation of slow motion. The external excitation is transformed unexpectedly. It has become proportional to the first derivative of the product of the slow variable amplitudes of both external and parametric high frequency excitations.

The properties of the averaged system are illustrated in the following figures, obtained by numeric simulation of the full equation (7.64). Figure 7.5 shows the solution for the parameter's set: $\alpha = 0;\quad \omega = 100;\quad \theta = 0.$

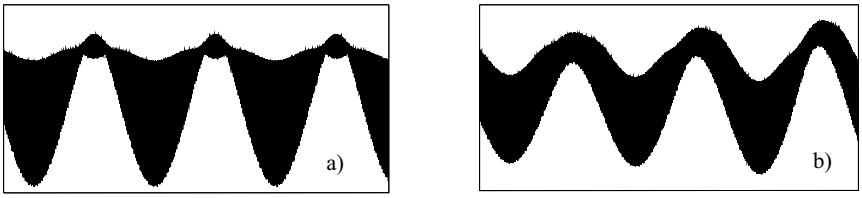

Fig. 7.5. System with strong excitation; velocity; a) modulation which does not excite the slow resonance: $f = 1 + 0.5\sin t; \quad c = \frac{1}{3}(1 + 0.5\sin t)^{-1}$; b) modulation exciting the slow resonance: $f = 1; \quad c = \frac{1}{3}(1 + 0.5\sin t)^{-1}$

This figure illustrates the typical character of the solutions to the systems with strong excitation, which is a superposition of slow motion and fast oscillations. The amplitude of the fast velocity oscillations is comparable with the amplitude of its slow evolution.

In case a) we have $fc = const \Rightarrow \dot{f}c + f\dot{c} = 0$. As it can be seen, in this case we have stationary oscillations of the averaged system. Figure 7.5 b) shows the results of the simulation for the case $fc \neq const \Rightarrow \dot{f}c + f\dot{c} \neq 0$. In this case, according to the prediction, we have a typical picture of the external non parametric resonance with the linearly increasing amplitude of slow oscillations.

In these cases there are no visible differences between analytic and numeric predictions. Figure 7.6 shows the comparison of analytic and numeric solutions for $\omega = 5$. We can see that, although in this case the small parameter is not small enough, the asymptotic solution still gives the qualitative character of the system's movement. However, the quantitative differences are significant.

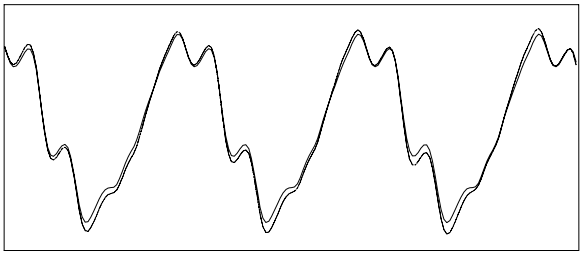

Fig. 7.6. Comparison of analytic and numeric predictions; travels

The performed analysis can be generalized for the case of the very strong external excitation. The general analysis of the systems with the very strong excitation still remains unaccomplished yet. But already this particular case demonstrates some significant peculiarities of the very strong excitation.

7.5 Systems with Very strong Excitation in the Special Case of Fast Oscillating Inertial Coefficients

Now we are going to consider systems with very strong excitation. Unfortunately it is not possible to perform this analysis for an arbitrary function Φ. That's why we restrict it to the special form:

$$\begin{aligned} &\ddot{x} = \omega^2 \Phi_0(t,\tau) + \omega \Phi_1(t,\tau)\dot{x} + F_1(t,\tau)\dot{x} + F_2(x,t,\tau) \\ &x(0) = x_0;\ \ \dot{x}(0) = v_0;\ \ \tau = \omega t \end{aligned} \tag{7.75}$$

Such a system appears naturally in mechanics, if the corresponding kinetic energy contains inertial coefficients depending on the fast time: $T = \frac{1}{2}\dot{x}^T J(t,\tau)\dot{x}$ and the external excitation is very strong.

The analysis of this system is from the technical point of view very similar to the case of the strong excitation. It would be natural to expect "*the strong response*", i.e. solutions with $x >> O(1)$, for systems with very strong excitation. But under some special conditions the limited solutions $x = O(1)$ are also possible. We will call these solutions "*the weak response*".

The equations governing the slow motions can be formally obtained for the system (7.75) using both the multiple scales and the modified averaging method. Let us apply the first one.

We convert as usual to an equation with partial derivatives which we try to solve asymptotically (compare with the equations (7.6), (7.7)). Balancing the terms of the same magnitude order one obtains the following equations:

$$\omega^2 : \frac{\partial^2 \psi_0}{\partial \tau^2} = \Phi_1 \frac{\partial \psi_0}{\partial \tau} + \Phi_0 \tag{7.76}$$

$$\omega^1 : \frac{\partial^2 \psi_1}{\partial \tau^2} + 2\frac{\partial^2 \psi_0}{\partial t \partial \tau} = \Phi_1 \left(\frac{\partial \psi_1}{\partial \tau} + \frac{\partial \psi_0}{\partial t} \right) + F_1 \frac{\partial \psi_0}{\partial \tau} \tag{7.77}$$

$$\begin{aligned} \omega^0 : &\frac{\partial^2 \psi_2}{\partial \tau^2} + 2\frac{\partial^2 \psi_1}{\partial t \partial \tau} + \frac{\partial^2 \psi_0}{\partial t^2} \\ &= \Phi_1 \left(\frac{\partial \psi_2}{\partial \tau} + \frac{\partial \psi_1}{\partial t} \right) + F_1 \left(\frac{\partial \psi_1}{\partial \tau} + \frac{\partial \psi_0}{\partial t} \right) + F_2(\psi_0, t, \tau) \end{aligned} \tag{7.78}$$

Let us try to find a solution to these equations as a superposition of the slow evolution and fast oscillations being of the same order of magnitude already at the coordinates' level. Let us rewrite the equation (7.76) as follows:

$$\frac{\partial u_0}{\partial \tau} = \Phi_1 u_0 + \Phi_0;\ u_0 = \frac{\partial \psi_0}{\partial \tau} \tag{7.79}$$

We require that this equation has a periodic solution with vanishing average; otherwise the function ψ_0 would increase with the fast time. It is possible only if the following condition is fulfilled

$$\left\langle W_*^T \Phi_0 \right\rangle = 0 \tag{7.80}$$

Here W_* is the fundamental matrix of the homogeneous system conjugated to (7.79):

$$\frac{\partial W_*}{\partial \tau} = -\Phi_1^T W_*;\ \ \det\left\langle W_*^T \right\rangle \neq 0 \tag{7.81}$$

If such a solution is found one can represent the corresponding solution to (7.76) as follows:

$$\psi_0 = X_0(t) + \Psi(t,\tau);\ \Psi(t,\tau) = \int_0^\tau \left(u_0 - \dot{X}_0\right) d\tilde{\tau} \tag{7.82}$$

The equation (7.77) can be analyzed similarly. We can transform it:

$$\frac{\partial u_1}{\partial \tau} = \Phi_1 u_1 + F_1 u_0 - \frac{\partial u_0}{\partial t};\ u_1 = \frac{\partial \psi_1}{\partial \tau} + \frac{\partial \psi_0}{\partial t} \tag{7.83}$$

We require that this equation has a periodic solution satisfying the condition

$$\left\langle u_1(t,\tau) \right\rangle = \dot{X}_0 \tag{7.84}$$

It is only possible if the following condition is fulfilled:

$$\left\langle W_*^T \left(F_1 u_0 - \partial u_0 / \partial t\right) \right\rangle = 0 \tag{7.85}$$

Finally we can consider the equation (7.78). It can be transformed to the form:

$$\begin{aligned} &\frac{\partial u_2}{\partial \tau} = \Phi_1 u_2 + F_1\left(u_1 + \dot{X}_0\right) - \frac{\partial u_1}{\partial t} + F_2\left(X_0 + \Psi, t, \tau\right) - \ddot{X}_0 \\ &u_2 = \frac{\partial \psi_2}{\partial \tau} + \frac{\partial \psi_1}{\partial t} \end{aligned} \tag{7.86}$$

The requirement that this equation must have a periodic solution with respect to the fast time, leads us directly to the equation governing the slow motion which can be written down as follows:

$$\begin{aligned} &M\left(X_0,\dot{X}_0,t\right)\ddot{X}_0 = V\left(X_0,\dot{X}_0,t\right);\\ &M(X_0,\dot{X}_0,t) = \left\langle W_*^T\right\rangle;\\ &V(X_0,\dot{X}_0,t) = \left\langle W_*^T F_1(t,\tau)\right\rangle \dot{X}_0 + \left\langle W_*^T F\left(X_0+\Psi,t,\tau\right)\right\rangle\\ &\qquad + \left\langle W_*^T\left(F_1 u_1 - \frac{\partial u_1}{\partial t}\right)\right\rangle;\\ &X_0\big|_{t=0} = x_0 - \Psi\big|_{t=0,\tau=0}, \dot{X}_0\big|_{t=0} = v_0 - u_1 - \omega u_0\big|_{t=0,\tau=0} \end{aligned} \tag{7.87}$$

The last term in the function V also contains the slow acceleration $\ddot{X}_0$.

Under certain mathematical conditions concerning the continuity of the involved functions, the solutions to the systems (7.87) and (7.75) are close to each other.

It is important to notice that the slow component of the solution in this case is overlapped by fast oscillations of the same order of magnitude. The slow component of the velocity is overlapped by the fast oscillations which are much larger than the slow component. The next examples illustrate this general result.

7.6 Response of a One Degree of Freedom Nonlinear System to Very Strong HF External and Strong Parametric Excitation due to Oscillating Inertia

We are going now to consider the same system as in the section 7.4 but with very strong external excitation:

$$\frac{d}{dt}\left(\frac{\dot{x}}{1+c(t)\cos\omega t}\right) + x + \alpha x^3 = \omega^2 f(t)\sin\left(\omega t+\theta\right) \tag{7.88}$$

This equation can be rewritten as follows:

$$\begin{aligned} \ddot{x} = {} & \omega^2 f\sin\left(\tau+\theta\right)\left(1+c\cos\tau\right) - \omega\frac{\dot{x}c\sin\tau}{1+c\cos\tau}\\ & + \frac{\dot{x}\dot{c}\cos\tau}{1+c\cos\tau} - \left(x+\alpha x^3\right)\left(1+c\cos\tau\right) \end{aligned} \tag{7.89}$$

7.6.1 Obtaining the Equations Governing the Slow Motion

Referring back to the general form (7.75) we can use the following notation:

$$\Phi_0 = f\sin(\tau+\theta)(1+c\cos\tau);\ \ \Phi_1 = -\frac{c\sin\tau}{1+c\cos\tau}$$
$$F_1 = \frac{\dot{c}\cos\tau}{1+c\cos\tau}; F_2 = -\left(x+\alpha x^3\right)(1+c\cos\tau) \tag{7.90}$$

The corresponding equation of fast motion (7.79) has the following form:

$$\frac{\partial u_0}{\partial\tau} = f\sin(\tau+\theta)(1+c\cos\tau) - \frac{c\sin\tau}{1+c\cos\tau}u_0 \tag{7.91}$$

This equation does not differ from that already solved equation (7.70). The only difference is that now its average has to vanish, i.e. $\langle u\rangle = 0$. The corresponding solution is:

$$u_0 = \left(\frac{1}{2}fc\cos\theta - f\cos(\tau+\theta)\right)(1+c\cos\tau)$$
$$\Psi = -f\sin(\tau+\theta) - \frac{1}{4}fc\sin(2\tau+\theta) + \frac{1}{2}fc^2\sin\tau\cos\theta. \tag{7.92}$$

Equation (7.83) can be rewritten as follows:

$$\frac{\partial u_1}{\partial\tau} = -\frac{u_1 c\sin\tau}{1+c\cos\tau} - (1+c\cos\tau)\frac{\partial}{\partial t}\left(\frac{u_0}{1+c\cos\tau}\right) \tag{7.93}$$

This equation does not contain X_0. Consequently this equation cannot be used in order to determine the slow part of the solution. However, it gives us a restriction, which must be fulfilled by functions $f(t)$ and $c(t)$. This restriction is necessary for the solutions of this type to exist. This condition from the mathematical point of view means that we require the periodicity of the solutions to (7.93). It has the form (7.85):

$$\left(\dot{f}c + f\dot{c}\right)\cos\theta = 0 \tag{7.94}$$

Assuming, that this condition is fulfilled, we can find the function u_1 satisfying the averaging condition $\langle u_1\rangle = \dot{X}_0$:

$$u_1 = \left(\dot{X}_0 - \tfrac{1}{2}\dot{f}c\sin\theta + \dot{f}\sin\left(\tau+\theta\right)\right)\left(1+c\cos\tau\right) \tag{7.95}$$

Finally, averaging the corresponding terms according to (7.87), we obtain the equation of slow motion:

$$\begin{aligned}\ddot{X}_0 + \left\{1+\tfrac{3}{2}\alpha f^2\left(1+\tfrac{1}{16}c^2+c^2\cos^2\theta\left(\tfrac{1}{4}c^2-1\right)\right)\right\}X_0 + \alpha X_0^{\,3} \\ = \tfrac{1}{2}\left(\ddot{f}c+\dot{f}\dot{c}\right)\sin\theta + \tfrac{3}{16}\alpha f^3 c\sin\theta\left(1-\tfrac{1}{4}c^4\cos^2\theta\right)\end{aligned} \tag{7.96}$$

7.6.2 Discussion of the Results

The main properties of this system are very similar to those of the system with strong excitation. The frequency of the free oscillations of the averaged system depends on the amplitude of the high frequency excitation. If both external and parametric excitations are slowly modulated, we can find both parametric and external slow excitations in the averaged system. However, they are significantly changed. In order for slow parametric excitation to exist, the system has to be nonlinear ($\alpha \neq 0$) and it is also necessary to have a modulated external excitation ($f \neq 0$).

If we consider the simplest linear situation, we obtain an equation, which is very similar to the situation of the strong excitation:

$$\ddot{X}_0 + X_0 = \tfrac{1}{2}\left(\ddot{f}c+\dot{f}\dot{c}\right)\sin\theta \tag{7.97}$$

The only difference is that the external slow excitation depends on the second derivative of the slow modulation. Another point is that in this case the excitation is proportional to the sine of the phase difference between the external and the parametric high frequency excitations and not to the cosine as in the previous case.

However we should not forget the relationship (7.94). It couples the modulating functions if $\cos\theta \neq 0$.

$$\dot{f}c + f\dot{c} = 0 \Rightarrow c(t) = \frac{c_0}{f(t)} \tag{7.98}$$

Then the equation (7.97) can be rewritten as follows:

$$\ddot{X}_0 + X_0 = \tfrac{1}{2}\frac{c_0}{f^2}\left(\ddot{f}f - \dot{f}^2\right)\sin\theta \tag{7.99}$$

It should be noticed, that a particular example of this system, without parametric excitation, was considered by Nayfeh and Nayfeh [81]. Equation (7.96) conforms completely to their results.

Figure 7.7 represents the direct numeric simulation results with regard to the full equations (5.3.1). The calculation was carried out using the following parameters: $\alpha = 0; \quad \theta = 0; \quad \omega = 100; \quad f = 1 + 0.5\sin t; \quad c = \dfrac{1}{3\left(1 + 0.5\sin t\right)}$

It can be noticed, that the solution now is a superposition of fast oscillations and comparatively large slow motions – even at the level of the generalized coordinates.

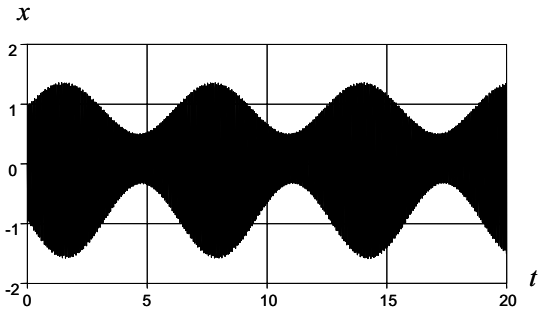

Fig. 7.7. Weak response in a system with very strong excitation

7.6.3 Large Solutions

At the beginning of this analysis we have assumed, that the necessary condition for the existence of such solutions (7.94) is fulfilled. But the equation can be also analyzed presumed this condition is unfulfilled. A solution with the amplitude order 1 does not exist in this case. However, there are solutions with larger amplitudes. In order to demonstrate this, the simplest linear situation will be considered and the scale of the variable x changed:

$$x = \omega z \tag{7.100}$$

The new variable z is governed by the following equation:

$$\begin{aligned} \ddot{z} = &-z\left(1 + c\cos\tau\right) + \frac{\dot{z}\dot{c}\cos\tau}{1 + c\cos\tau} \\ &+ \omega f \sin\left(\tau + \theta\right)\left(1 + c\cos\tau\right) - \omega\frac{\dot{z}c\sin\tau}{1 + c\cos\tau} \end{aligned} \tag{7.101}$$

This is the linear variant of the previously analyzed equation with strong excitation (7.68). Its solutions are known.

Hence, in systems with very strong excitation we should distinguish between two types of solutions. These will be referred to as “the weak response” and “the strong response”. The strong response exists under more general conditions than the weak response. The slow amplitude of the strong response is significantly larger than 1. If certain additional restrictions on the character of the excitation’s modulation are fulfilled, the weak response appears alongside with the strong response. Its amplitude has the order of magnitude 1.

Figure 7.8 represents the direct numeric simulation results with regard to the full equation (7.88). The calculation was carried out using the following parameter’s values: $\alpha = 0; \quad \theta = 0; \quad \omega = 100; \quad f = 1; \quad c = \dfrac{1}{3\left(1+0.5\sin t\right)}$

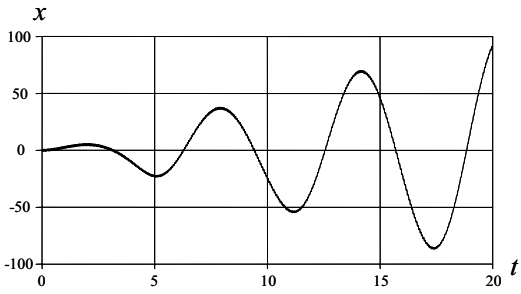

Fig. 7.8. Strong response in a system with very strong excitation

In this situation the weak response does not exist. The reaction of the system is typical for the external resonance, as predicted by the equation of the slow motion (7.74). The amplitude of the oscillations has from the very beginning the magnitude order ω.

7.7 Dynamics of a Two Link Pendulum with a Fast Rotating Second Link

A two link pendulum is considered as the last example in this section in order to demonstrate how the equations with very strong external excitation and strong parametric excitation depending on the velocity appear in mechanical systems.

7.7.1 Equations of Motion and Their Transformation to the Basic Form for Systems with Very Strong Excitation

The system consists of the first link characterized by its mass M and inertia J_1 and the second link of the mass m and inertia J_2 suspended at the inertia axis of the first link (*cf.* Fig. 7.9).

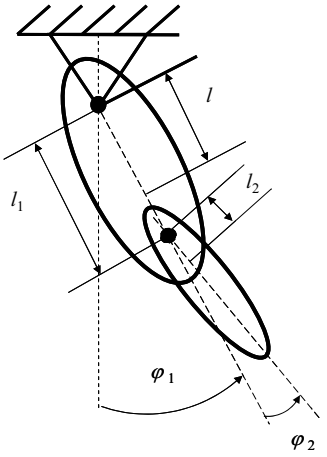

Fig. 7.9. The two link pendulum

The distance between the suspension points of the links is l_1, the distance between the suspension point and the mass point of the first link is l, and between the corresponding suspension point and the mass point of the second link is l_2. The generalized coordinates of the links φ_1 and φ_2 are shown in Fig. 7.9.

The equations of motion for this two link mechanism can be written as follows (*cf.* [117]):

$$\begin{aligned}&\left(J_1 + ml_1^2 + J_2 + 2ml_1l_2\cos\varphi_2\right)\ddot{\varphi}_1 + \left(J_2 + ml_1l_2\cos\varphi_2\right)\ddot{\varphi}_2 \\ &\qquad -2ml_1l_2\sin\varphi_2\dot{\varphi}_1\dot{\varphi}_2 - ml_1l_2\sin\varphi_2\dot{\varphi}_2^{\,2} = M_1 \\ &J_2\ddot{\varphi}_2 + \left(J_2 + ml_1l_2\cos\varphi_2\right)\ddot{\varphi}_1 - ml_1l_2\sin\varphi_2\dot{\varphi}_1\dot{\varphi}_2 = M_2\end{aligned} \tag{7.102}$$

M_1 and M_2 are the external torques. We assume that the first external torque is caused by the gravity:

$$M_1 = -Mgl\sin\varphi_1 - mgl_2\sin\left(\varphi_1 + \varphi_2\right) \tag{7.103}$$

There is an electric drive in the junction between the first and the second link which holds the rotation speed of the second link constant:

$$\dot{\varphi}_2 = \omega = const \tag{7.104}$$

This is a simple model representing a fast rotating unbalanced mechanism attached to some nonlinear flexible structure. Our aim now is to show how the fast motion of the second link can influence the slow properties of the flexible base. Inserting (7.104) into the first equation (7.102) one obtains the following equation:

$$\ddot{\varphi} = \frac{\omega^2 \sin \omega t + 2\omega\dot{\varphi}\sin \omega t - k^2\left(\mu \sin \varphi + \sin\left(\omega t + \varphi\right)\right)}{\rho + 2\cos \omega t} \tag{7.105}$$

The following notation is used here:

$$\mu = \frac{Ml}{ml_2};\;\; L = l_1\sqrt{\frac{J_1 + J_2}{ml_1^2}};\;\; \rho = \frac{L^2}{l_1 l_2} > 2;\;\; k = \sqrt{\frac{g}{l_1}}$$
$$\varphi = \varphi_1 \tag{7.106}$$

This is a system with the very strong external excitation if the following assumptions concerning the magnitude orders of the parameters are fulfilled: $\omega >> 1;\; k, \mu, \rho = O(1)$.

7.7.2 Obtaining Equations Governing the Slow Motion

Referring back to the general form (7.75) one can use the following notation:

$$\Phi_0 = \frac{\sin \tau}{\rho + 2\cos \tau};\;\; \Phi_1 = \frac{2\sin \tau}{\rho + 2\cos \tau}$$
$$F_1 = 0;\;\; F_2 = -\frac{k^2\left(\mu \sin \varphi + \sin\left(\tau + \varphi\right)\right)}{\rho + 2\cos \tau};\;\; \tau = \omega t \tag{7.107}$$

The very strong external excitation appears as a natural result of the fast rotating unbalanced second link. The strong excitation depending on the velocity is the result of the variable effective inertia of the pendulum as a whole.

We are interested in the limited solutions to the equation (7.105), i.e. in the oscillations of the pendulum corresponding to the weak response due to the terminology of the previous section. The strong response corresponds then to the rotational motion.

Equation of the fast motion (7.79) can be written as follows:

$$\frac{\partial u_0}{\partial \tau} = \frac{2\sin \tau}{\rho + \cos \tau}\left(u_0 + \frac{1}{2}\right) \tag{7.108}$$

Its solution satisfying the periodicity condition for the fast oscillating motion is:

$$u_0 = \frac{1}{2}\left(\frac{\sqrt{\rho^2 - 4}}{\rho + 2\cos\tau} - 1 \right) \tag{7.109}$$

The equation conjugated to the homogeneous one is

$$\frac{\partial W_*}{\partial \tau} = -W_* \frac{2\sin\tau}{\rho + 2\cos\tau} \quad \Rightarrow \quad W_* = \rho + 2\cos\tau \tag{7.110}$$

Now the equation (7.83) has to be considered. The necessary condition for the existence of its periodic solution is fulfilled automatically ($F_1 = 0;\ \partial u_0 / \partial t = 0$). The equation itself takes the form:

$$\frac{\partial u_1}{\partial \tau} = \frac{2\sin\tau}{\rho + 2\cos\tau} u_1;\ \ \langle u_1 \rangle = \dot{X}_0 \tag{7.111}$$

Its solution is:

$$u_1 = \dot{X}_0 \left(\frac{\sqrt{\rho^2 - 4}}{\rho + 2\cos\tau} - 1 \right) = 2\dot{X}_0 u_0 \tag{7.112}$$

In order to obtain the equation of slow motion we have now to average the corresponding terms. The function Ψ describing the large oscillating part of the solution must be calculated first.

$$\begin{aligned} \Psi &= \frac{1}{2}\int_0^{\tau} \left(\frac{\sqrt{\rho^2 - 4}}{\rho + 2\cos\tilde{\tau}} - 1 \right) d\tilde{\tau} \\ &= -\frac{\tau}{2} + \begin{cases} n\pi + \arctan\left(\sqrt{\dfrac{\rho - 2}{\rho + 2}} \tan\dfrac{\tau}{2} \right) \\ n\pi \le \tau < (n+1)\pi;\ \ n = 0, 1, 2, \ldots \end{cases} \end{aligned} \tag{7.113}$$

This function has several interesting properties:

1. $\Psi(\tau) + \Psi(2\pi - \tau) = 0$
2. $\langle \sin\Psi(\tau) \rangle = 0$
3. $\langle \sin(\Psi(\tau) + \tau) \rangle = 0$

4. $\left\langle \cos\Psi(\tau)\right\rangle = \frac{1}{\pi}\int_0^{\pi}\cos\Psi(\tau)\,d\tau$

5. $\left\langle \cos\left(\Psi(\tau)+\tau\right)\right\rangle = \frac{1}{\pi}\int_0^{\pi}\cos\left(\Psi(\tau)+\tau\right)d\tau$

Taking (7.112) into account that one can notice that

$$\frac{\partial v}{\partial t} = 2\ddot{X}_0 u \tag{7.114}$$

The equation of slow motion can be rewritten down as follows:

$$\sqrt{\rho^2-4}\,\ddot{X}_0 = -k^2\left(\mu\left\langle\cos\left(\Psi(\tau)\right)\right\rangle + \left\langle\cos\left(\Psi(\tau)+\tau\right)\right\rangle\right)\sin X_0 \tag{7.115}$$

It is not a trivial problem to calculate the required averages. They cannot be expressed in elementary function. However it can be done using the full elliptic integrals of the first and the second art:

$$E(x) = \int_0^{\frac{\pi}{2}}\sqrt{1-x^2\sin^2\tau}\;\;d\tau;\;\; K(x) = \int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1-x^2\sin^2\tau}}\;\;d\tau \tag{7.116}$$

Then the following relationships are valid:

$$\begin{aligned}
\left\langle\cos\left(\Psi(\tau)\right)\right\rangle &= \frac{1}{\pi}\int_0^{\pi}\cos\left\{\arctan\left(\sqrt{\frac{\rho-2}{\rho+2}}\tan\frac{\tau}{2}\right)-\frac{\tau}{2}\right\}d\tau = \\
&= \frac{2}{\pi}\frac{E\left(\frac{2}{\sqrt{\rho+2}}\right)+\sqrt{\frac{\rho-2}{\rho+2}}K\left(\frac{2}{\sqrt{\rho+2}}\right)}{1+\sqrt{(\rho-2)/(\rho+2)}} = S_0(\rho) \\
\left\langle\cos\left(\Psi(\tau)+\tau\right)\right\rangle &= \frac{1}{\pi}\int_0^{\pi}\cos\left\{\arctan\left(\sqrt{\frac{\rho-2}{\rho+2}}\tan\frac{\tau}{2}\right)+\frac{\tau}{2}\right\}d\tau = \\
&= \frac{2}{\pi}\frac{E\left(\frac{2}{\sqrt{\rho+2}}\right)-\sqrt{\frac{\rho-2}{\rho+2}}K\left(\frac{2}{\sqrt{\rho+2}}\right)}{1-\sqrt{(\rho-2)/(\rho+2)}} = S_1(\rho)
\end{aligned} \tag{7.117}$$

The alternative way is to tabulate these functions numerically. The numerical results are shown in Figure 7.10.

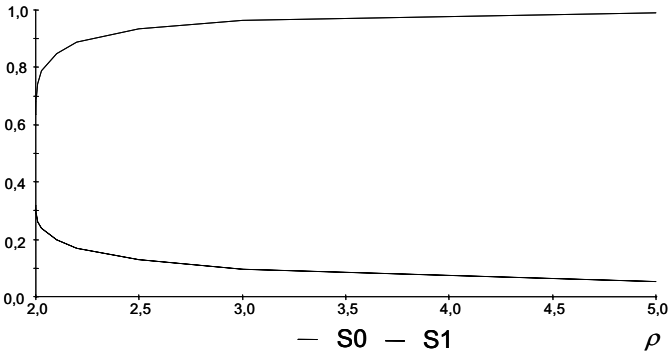

Fig. 7.10. Functions S_0 and S_1

The equation of slow motion for the pendulum takes its final form if we use these functions:

$$\sqrt{\rho^2-4}\ddot{X}_0 + k^2\left(\mu S_0(\rho) + S_1(\rho)\right)\sin X_0 = 0 \tag{7.118}$$

7.7.3 Discussion of the Results

A typical numerical solution of the full equation (7.105) for the case of the limited slow oscillations of the pendulum is shown in Fig. 7.11. The simulations were performed for the following parameter's values:

$\rho = 2,5; \quad k = 1; \quad \mu = 1; \quad \omega = 10.$

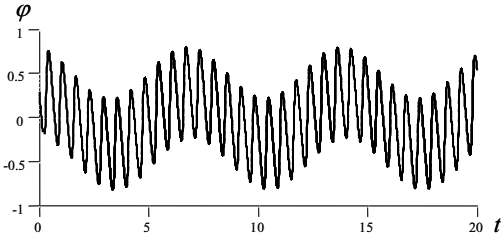

Fig. 7.11. Small slow oscillations of the pendulum

It is easy to notice here that the solution is a superposition of slow and fast oscillations of the same magnitude order at the level of the coordinates, which is characteristic for systems with very strong excitation.

Fig. 7.12 shows the same simulation but for different initial conditions corresponding to the motion in the vicinity of the heteroclinic curves.

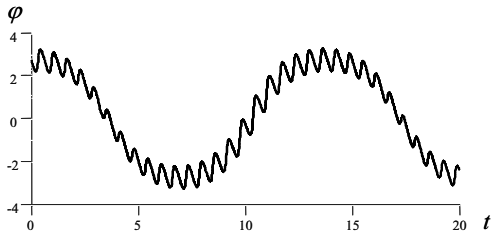

Fig. 7.12. Large slow oscillations of the pendulum

Figure 7.13 displays the slow rotations of the pendulum for the corresponding initial conditions.

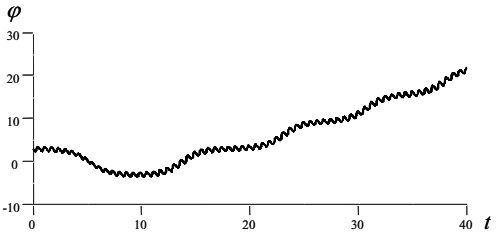

Fig. 7.13. Slow rotation of the pendulum

Equation (7.118) shows that the "slow" oscillation's frequency of the pendulum increases as $\rho \to 2$. The corresponding comparison between the theoretic and numeric results is shown in Figure 7.14.

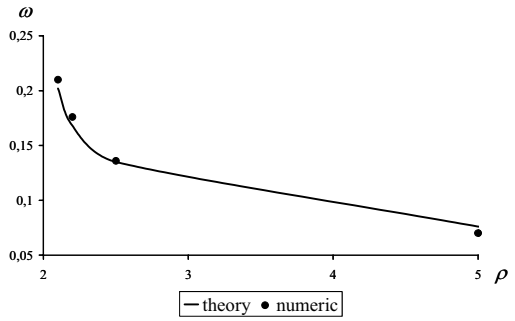

Fig. 7.14. Frequency of the "small" slow oscillations as a function of the parameter ρ; comparison between approximate and numeric solutions

7.7.4 A Short Remark on the Practical Importance of the Considered Solutions

An interesting problem is connected with the existence area of the considered type of motion in the initial conditions space. The initial conditions for the slow variable are determined according to (7.87):

$$X_0(0) = \varphi_0$$
$$\dot{X}_0(0) = \dot{\varphi}_0\sqrt{\frac{\rho+2}{\rho-2}} + \frac{1}{2}\omega\left(\sqrt{\frac{\rho+2}{\rho-2}} - 1\right) \tag{7.119}$$

These values (first of all for $\dot{X}_0$) must be of the order of magnitude 1. This requirement leads to the desired condition:

$$\dot{\varphi}_0 = -\frac{1}{2}\omega\left(1 - \sqrt{\frac{\rho-2}{\rho+2}}\right) + O(1) \tag{7.120}$$

The relationship (7.120) seems to be a very special condition. The initial velocity must be inside of a relative small $O(1)$ strip around the large value $O(\omega)$. The fast rotation of the pendulum is the solution for all other initial conditions. This rotation corresponds to the strong response in the terminology of the previous section. In other words the attraction area of the weak response is mach smaller that the attraction area of the strong response.

So the importance of the considered limited solution seems to be relative small. However exactly this type of motion is excited by the straight forward acceleration of the second link from the rest to some large value. This interesting fact is illustrated by the numeric simulation results in Figure 7.15. The second pendulum's link was slowly accelerated so that its rotation speed increased according the following relationship:

$$\omega = 10\left(1 - e^{-0{,}15\,t}\right) \tag{7.121}$$

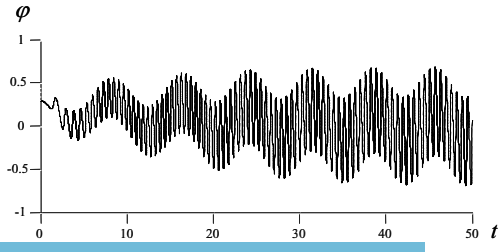

Fig. 7.15. Oscillations of the pendulum during and after the acceleration of the second link

From this point of view the limited solutions in the considered system are important for the practice.

> Remark: Systems with dominating terms depending on the velocities were considered in this chapter. One could try to apply this approach to the control problem investigated in the previous chapter. It is however not possible, because the corresponding equation of the fast motion does not have the general periodic solution which is a significant requirement for the approach discussed in this chapter.

7.8 Conclusions

The general analysis of systems with strong HF excitation was performed in this chapter. The main attention was attracted to the systems with HF terms depending on the generalized velocities. The following general properties of the solutions to these systems can be emphasized.

- It is typical for systems with strong excitation that their solution is a superposition of a slow component and fast oscillations. These are small respectively to the generalized coordinates x, but their derivatives are not small respectively to the generalized velocities $\dot{x}$.
- The main difference between systems with the HF excitation depending on the velocities and those with the HF excitation depending only on the positional coordinates can be formulated very simply:

> The HF excitation depending on the velocities changes the effective mass of the system with respect to slow motions.

The performed analysis is illustrated by several mathematical and mechanical examples. It is also generalized for systems with very strong external excitation which doesn't depend on the state variables. In this case the fast vibrations are not small even at the coordinates' level. However limited solutions are possible also in these systems. The dynamics of a two link pendulum with the fast rotating second link is considered as the mechanical example. It illustrates the main dynamic effects of the very strong HF excitation on the light flexible structures carrying unbalanced rotors, for which the performed analysis is especially important.

Appendixes

Appendix I: The first Bogoliubov's Theorem for Standard Averaging

Specific Gronwall's lemma

Consider a linear integral inequality

$$w(t) \le a + b\int_{t_0}^{t} w(\tau)\,d\tau \tag{I.1}$$

Suppose

$$w(t) \ge 0,\ a > 0,\ b > 0 \tag{I.2}$$

Then the following inequality is also fulfilled:

$$w(t) \le a e^{b(t-t_0)} \tag{I.3}$$

Proof

The inequality (I.1) can be rewritten as follows

$$\frac{bw(t)}{a + b\int_{t_0}^{t} w(\tau)\,d\tau} \le b \tag{I.4}$$

Integrating it with respect to t one obtains:

$$\log\left(a + b\int_{t_0}^{t} w(\tau)\,d\tau\right) - \log a \le b(t - t_0) \tag{I.5}$$

Taking into account the monotony of the exponent we notice

$$a + b\int_{t_0}^{t} w(\tau)\,d\tau \le a e^{b(t-t_0)} \tag{I.6}$$

Applying the original inequality (I.1) we conclude:

$$w(t) \le a e^{b(t-t_0)} \tag{I.7}$$

This is the inequality we wanted to prove.

Theorem
Consider a system in the standard form

$$\dot{x} = \varepsilon X(x,t,\varepsilon), \qquad x(0) = x_0 \tag{I.8}$$

Suppose:

1. The vector function $X : x \in D \subset R^n, t \in R^1, \varepsilon << 1$ is a measurable T-periodic function with respect to t for $x \in D$

$$X(x, t+T, \varepsilon) = X(x,t,\varepsilon) \;\; \forall t, x \in D \tag{I.9}$$

2. It is bounded and satisfies Lipschitz-condition with respect to the vector-argument x:

$$\begin{aligned} &\left\| X(x,t,\varepsilon) \right\| \le M \\ &\left\| X(x_1,t,\varepsilon) - X(x_2,t,\varepsilon) \right\| \le \lambda \left\| x_1 - x_2 \right\| \end{aligned} \tag{I.10}$$

3. The time average of the function X exists uniformly with respect to x:

$$\left\langle X(x,t,\varepsilon) \right\rangle_t = \frac{1}{T}\int_0^T X(x,t,\varepsilon)dt \tag{I.11}$$

Consider the averaged system satisfying the same initial conditions alongside (I.8):

$$\begin{aligned} &\dot{\xi} = \varepsilon \Xi(\xi,\varepsilon), \quad \xi(0) = x_0 \\ &\Xi(\xi,\varepsilon) = \left\langle X(\xi,t,\varepsilon) \right\rangle_t \end{aligned} \tag{I.12}$$

Under these assumptions the difference between the solutions of the original system (I.8) and the averaged system (I.12) can be estimated as follows:

$$\left\| x - \xi \right\| \le C_1 \varepsilon e^{\varepsilon C_2 t} \tag{I.13}$$

The constants C_1 and C_2 does not depend on the small parameter ε.

Proof

(The following proof is based on [114, 139])

The difference between the solutions to (I.8) and (I.12) can be estimated as follows:

$$\begin{aligned}\|x-\xi\| &\le \varepsilon\left\|\int_0^t\left(X\left(x,t,\varepsilon\right)-\Xi\left(\xi,\varepsilon\right)\right)dt\right\| \\ &\le \varepsilon\left\|\int_0^t\left(X\left(x,t,\varepsilon\right)-X\left(\xi,t,\varepsilon\right)\right)dt\right\| \\ &+\varepsilon\left\|\int_0^t\left(X\left(\xi,t,\varepsilon\right)-\Xi\left(\xi,\varepsilon\right)\right)dt\right\|\end{aligned} \tag{I.14}$$

According to Lipschitz-condition (I.10) the following estimation can be obtained from this inequality:

$$\|x-\xi\| \le \varepsilon\lambda\int_0^t\|x.-\xi\|dt+\varepsilon\left\|\int_0^{(N+1)T}\left(X\left(\xi,t,\varepsilon\right)-\Xi\left(\xi,\varepsilon\right)\right)dt\right\|, \tag{I.15}$$

$$N=\left[t/T\right], \text{i.e. } NT\le t<\left(N+1\right)T$$

In order to estimate the second term the values $\xi_j=\xi\left(jT\right)$ can be introduced. Notice that the following identity is valid because Ξ is the average of X:

$$\left\|\int_{jT}^{(j+1)T}\left(X\left(\xi_j,t,\varepsilon\right)-\Xi\left(\xi_j,\varepsilon\right)\right)dt\right\|=0,\ j=1,2,3,\ldots \tag{I.16}$$

Then we can estimate:

$$\begin{aligned}\|x-\xi\| &\le \varepsilon\lambda\int_0^t \|x.-\xi\|dt \\ &+\varepsilon\sum_{j=0}^{N}\left\|\int_{jT}^{(j+1)T}\left(X(\xi,t,\varepsilon)-X(\xi_j,t,\varepsilon)\right)dt\right\| \\ &+\varepsilon\sum_{j=0}^{N}\left\|\int_{jT}^{(j+1)T}\left(\Xi(\xi,\varepsilon)-\Xi(\xi_j,\varepsilon)\right)dt\right\| \\ &\le \varepsilon\lambda\int_0^t \|x-\xi\|dt+2\varepsilon\lambda\sum_{j=0}^{N}\int_{jT}^{(j+1)T}\|\xi-\xi_j\|dt\end{aligned} \tag{I.17}$$

But ξ is the solution of the system

$$\dot{\xi}=\varepsilon\Xi(\xi,\varepsilon),\ \ \xi(jT)=\xi_j \tag{I.18}$$

Hence using (I.18) only for one period, one finds

$$\|\xi-\xi_j\|\le\varepsilon TM \tag{I.19}$$

Using (I.19) the following estimation for the accuracy of the first order averaging can be obtained:

$$\|x-\xi\|\le\varepsilon\lambda\int_0^t\|x-\xi\|dt+2\varepsilon^2\lambda T^2M(N+1) \tag{I.20}$$

The last step is to apply the Gronwall's lemma giving the final estimate:

$$\|x-\xi\|\le 2\lambda M\varepsilon^2T^2(N+1)e^{\varepsilon\lambda t} \tag{I.21}$$

This is the required result (I.13) with

$$\begin{aligned}C_1&=2\lambda MT\{\varepsilon(N+1)T\}\\ C_2&=\lambda\end{aligned} \tag{I.22}$$

It guarantees that the mistake by using the averaged system instead of the original one has the magnitude order of the small parameter for the asymptotically long time interval $t=O(1/\varepsilon)$.

Appendix II: On the Attractive Properties of the Asymptotically Stable Equilibrium of the Averaged System

Preliminary remarks

Consider a system

$$\begin{aligned} &\dot{\xi} = A\xi + \gamma(\xi, t),\ \gamma(0, t) = 0 \\ &\xi(t_0) = \xi_0 \end{aligned} \tag{II.1}$$

Here $\xi, \xi_0 \in D \subset R^n$. Suppose

1. A is a constant $n \times n$ matrix with eigenvalues having negative real parts only.
2. The function $\gamma(\xi, t)$ is continuous with respect to t.
3. The function $\gamma(\xi, t)$ is continuously differentiable with respect to ξ.
4. The function $\gamma(\xi, t) = o(\|\xi\|)$ as $\xi \to 0$, uniformly in t

Then there exist constants $t_0, a, b > 0$ such that if $\|\xi_0\| < a$ and $t > t_0$ then

$$\|\xi(t)\| \le \|\xi_0\| e^{-b(t-t_0)} \tag{II.2}$$

Even more: if we consider two solutions satisfying different initial conditions $\xi_1(t)$ and $\xi_2(t)$, then under the same conditions

$$\|\xi_1(t) - \xi_2(t)\| \le \|\xi_1(t_0) - \xi_2(t_0)\| e^{-b(t-t_0)} \tag{II.3}$$

This is the particular case of the classical theorem of Poincaré – Lyapunov. Its proof can be found in [114].

The following theorem describes the attractive properties of the asymptotically stable equilibrium of the averaged system.

Theorem (Eckhaus [29], Sanchez-Palencia [112])
Consider the initial value problem

$$\begin{aligned} &\dot{x} = \varepsilon X(x, t, \varepsilon),\ x(0) = x_0 \\ &x, x_0 \in D \subset R^n \end{aligned} \tag{II.4}$$

Suppose the function X satisfies all the conditions of the first Bogoliubov's theorem. Consider in addition the averaged system

$$\begin{aligned}\dot{\xi} &= \varepsilon \Xi\left(\xi,\varepsilon\right),\ \xi\left(0\right) = x_0 \\ \Xi\left(\xi,\varepsilon\right) &= \left\langle X\left(\xi,t,\varepsilon\right)\right\rangle_t\end{aligned} \tag{II.5}$$

Suppose:

1. This system has an equilibrium $\xi = 0$ which is asymptotically stable in the linear approximation.
2. The function $\Xi\left(\xi,\varepsilon\right)$ is continuously differentiable with respect to ξ in D.
3. The asymptotically stable equilibrium $\xi = 0$ has a domain of attraction $D_0 \subset D$.

Then if $x_0 \in D_0$ then

$$\left\|x\left(t\right) - \xi\left(t\right)\right\| \le C_3\varepsilon,\ 0 \le t < \infty \tag{II.6}$$

Proof

The following estimation is valid for an arbitrary $T = O\left(1\right)$ according to the first Bogoliubov's theorem:

$$\left\|x\left(t\right) - \xi\left(t\right)\right\| \le C_1\varepsilon e^{C_2 T},\ 0 \le t < \frac{T}{\varepsilon} \tag{II.7}$$

Let us split the time-axis in an infinite sequence of the long time intervals:

$$\left[0,\infty\right) = \left[0,\frac{T}{\varepsilon}\right] \cup \left[\frac{T}{\varepsilon},\frac{2T}{\varepsilon}\right] \cup \ldots \cup \left[\frac{nT}{\varepsilon},\frac{\left(n+1\right)T}{\varepsilon}\right] \cup \ldots \tag{II.8}$$

$$n = 1,2,\ldots$$

At each time interval we introduce the corresponding averaged system satisfying the correct initial conditions:

$$\dot{\xi}_n = \varepsilon \Xi\left(\xi_n,\varepsilon\right),\ \xi_n\left(\frac{nT}{\varepsilon}\right) = x\left(\frac{nT}{\varepsilon}\right) = x_n \tag{II.9}$$

According to (II.7) the following estimation is valid

$$\left\|x\left(t\right) - \xi_n\left(t\right)\right\| \le \delta = C_1\varepsilon e^{C_2 T},\ \frac{nT}{\varepsilon} \le t < \frac{\left(n+1\right)T}{\varepsilon} \tag{II.10}$$

On the other hand we can apply the inequality (II.3):

$$\left\|\xi(t)-\xi_n(t)\right\|_{\left[\frac{nT}{\varepsilon},\frac{(n+1)T}{\varepsilon}\right]} \le e^{-b\left(t-\frac{nT}{\varepsilon}\right)}\left\|\xi\left(\frac{nT}{\varepsilon}\right)-x_n\right\| \tag{II.11}$$

Notice that for the considered time interval

$$k = e^{-b\frac{T}{\varepsilon}} < 1 \tag{II.12}$$

Let us consider the first interval. The following estimates are valid:

$$\begin{aligned} &\left\|x(t)-\xi(t)\right\|_{\left[0,\frac{T}{\varepsilon}\right]} \le \delta,\ \xi_0(t)=\xi(t) \\ &\left\|x\left(\frac{T}{\varepsilon}\right)-\xi\left(\frac{T}{\varepsilon}\right)\right\| \le \delta \end{aligned} \tag{II.13}$$

Consider the second interval:

$$\begin{aligned} &\left\|x(t)-\xi(t)\right\|_{\left[\frac{T}{\varepsilon},\frac{2T}{\varepsilon}\right]} \le \left\|x(t)-\xi_1(t)\right\|_{\left[\frac{T}{\varepsilon},\frac{2T}{\varepsilon}\right]} + \left\|\xi_1(t)-\xi(t)\right\|_{\left[\frac{T}{\varepsilon},\frac{2T}{\varepsilon}\right]} \\ &\le \delta + e^{-b\left(t-\frac{T}{\varepsilon}\right)}\left\|x\left(\frac{T}{\varepsilon}\right)-\xi\left(\frac{T}{\varepsilon}\right)\right\| \le \delta\left(1+e^{-b\left(t-\frac{T}{\varepsilon}\right)}\right) \\ &\left\|x\left(\frac{2T}{\varepsilon}\right)-\xi\left(\frac{2T}{\varepsilon}\right)\right\| \le \delta(1+k) \end{aligned} \tag{II.14}$$

For the third interval we obtain

$$\begin{aligned} &\left\|x(t)-\xi(t)\right\|_{\left[\frac{2T}{\varepsilon},\frac{3T}{\varepsilon}\right]} \le \left\|x(t)-\xi_2(t)\right\|_{\left[\frac{2T}{\varepsilon},\frac{3T}{\varepsilon}\right]} \\ &\qquad + \left\|\xi_2(t)-\xi(t)\right\|_{\left[\frac{2T}{\varepsilon},\frac{3T}{\varepsilon}\right]} \\ &\le \delta + e^{-b\left(t-\frac{T}{\varepsilon}\right)}\left\|x\left(\frac{2T}{\varepsilon}\right)-\xi\left(\frac{2T}{\varepsilon}\right)\right\| \le \delta + \delta(1+k)e^{-b\left(t-\frac{2T}{\varepsilon}\right)} \\ &\left\|x\left(\frac{3T}{\varepsilon}\right)-\xi\left(\frac{3T}{\varepsilon}\right)\right\| \le \delta\left(1+k+k^2\right) \end{aligned} \tag{II.15}$$

Continuing this process we can easily show by induction that

$$\begin{aligned}&\left\|x(t)-\xi(t)\right\|_{\left[\frac{nT}{\varepsilon},\frac{(n+1)T}{\varepsilon}\right]}\le\delta+\delta\left(1+k+\ldots+k^{n-1}\right)e^{-b\left(t-\frac{(n+1)T}{\varepsilon}\right)}\\&\left\|x\left(\frac{(n+1)T}{\varepsilon}\right)-\xi\left(\frac{(n+1)T}{\varepsilon}\right)\right\|\le\delta\left(1+k+\ldots+k^{n}\right)\end{aligned}\tag{II.16}$$

Taking the limit for $n\to\infty$, which corresponds to $t\to\infty$, we obtain the final estimate

$$\begin{aligned}&\left(1+k+\ldots+k^{n}\right)\le\frac{1}{1-k},\ e^{-b\left(t-\frac{(n+1)T}{\varepsilon}\right)}\le 1\Rightarrow\\&\left\|x(t)-\xi(t)\right\|\le\frac{1+k}{1-k}\delta\end{aligned}\tag{II.17}$$

Taking (II.7) and (II.12) into account we find

$$C_3=C_1e^{C_2T}\frac{1+e^{-b\frac{T}{\varepsilon}}}{1-e^{-b\frac{T}{\varepsilon}}}\tag{II.18}$$

This relationship completes the proof.

Note

This theorem actually requires continuity and smoothness of the averaged system and its asymptotic accuracy. It is not really important if the original system is continuous or not. Thus this theorem can be directly generalized for all discontinuous problems considered in this book as soon as the asymptotic accuracy of the corresponding averaged system is proved.

Appendix III: Averaging of Systems with Short Strong Perturbations

Theorem

(The following theorem and its proof are based on [36]. An alternative approach based on differential inclusions can be found in for example in [95 – 97].)

Consider the initial value problem

$$\begin{aligned} &\dot{x} = \varepsilon X\left(x,t\right) + Z\left(x,t\right)\left[E\left(g\left(t\right) + \varepsilon f\left(x,t\right)\right) - E\left(g\left(t\right)\right)\right] \\ &x\left(0\right) = x_0 \end{aligned} \tag{III.1}$$

Here $E\left(t\right)$ is the "one step" function:

$$E(\dot{x}) = \begin{cases} 1 \text{ if } \dot{x} > 0 \\ 1/2 \text{ if } \dot{x} = 0; \\ 0 \text{ if } \dot{x} < 0 \end{cases} \quad \begin{aligned} &E(\dot{x}) + E(-\dot{x}) = 1 \\ &E(\dot{x}) - E(-\dot{x}) = \operatorname{sgn}(\dot{x}) \end{aligned} \tag{III.2}$$

Consider the corresponding averaged problem alongside (III.1):

$$\begin{aligned} &\dot{\xi} = \varepsilon \Xi\left(\xi\right) + \varepsilon \varsigma\left(\xi\right) \ , \ \ \xi\left(0\right) = x_0 \\ &\Xi\left(\xi\right) = \left\langle X\left(\xi,t\right)\right\rangle_t \\ &\varepsilon \varsigma\left(\xi\right) = \left\langle Z\left(\xi,t\right)\left[E\left(g\left(t\right) + \varepsilon f\left(\xi,t\right)\right) - E\left(g\left(t\right)\right)\right]\right\rangle_t \end{aligned} \tag{III.3}$$

Here

$$\begin{aligned} &X, Z : R^{n+1} \to R^n; \quad f : \ R^{n+1} \to R; \quad g : \ R \to R \\ &x, \xi, x_0 \in D \subset R^n, t \in \left[0, \infty\right), \varepsilon \in \left(0, \varepsilon_0\right] \end{aligned} \tag{III.4}$$

Suppose:

1. X, Z and f are measurable functions of t for constant x and ε.
2. In addition f is a piecewise differentiable function with respect to t, and its derivative is bounded:

$$\sup \left\| \frac{\partial f}{\partial t} \right\| \le F_t \tag{III.5}$$

3. All the functions are T-periodic with respect to t. $\left\langle \ \right\rangle_t$ means the time average:

$$\left\langle f(t)\right\rangle_t = \frac{1}{T}\int_0^T f(t)dt \tag{III.6}$$

4. X and Z are bounded Lipschitz-continuous functions in x on D:

$$\begin{aligned} &\|X(x,t)\| \le M_X; \quad \|X(x_1,t)-X(x_2,t)\| \le \lambda_X \|x_1-x_2\| \\ &\|Z(x,t)\| \le M_Z; \quad \|Z(x_1,t)-Z(x_2,t)\| \le \lambda_Z \|x_1-x_2\| \end{aligned} \tag{III.7}$$

5. f is a bounded Lipschitz-continuous function in x on D:

$$\|f(x,t)\| \le M_f; \quad \|f(x_1,t)-f(x_2,t)\| \le \lambda_f \|x_1-x_2\| \tag{III.8}$$

6. $g(t) \in C^{(1)}[0,\infty)$.

7. The equation $g(t)=0$ has m solutions $t_{i0}, i=1,\ldots,m$ for $t \in [0,T)$ and

$$|g'(t_{i0})| \ge G > 0 \tag{III.9}$$

8. All constants do not depend on ε, and ξ belongs to the interior subset of D on the time scale $1/\varepsilon$.

Then the solutions to (III.1) and (III.3) are asymptotically close to each other, i.e. the error one makes on using the averaged system instead of the original one is small for the asymptotically long time interval:

$$\|x(t)-\xi(t)\| \le C_1 \varepsilon e^{C_2 \varepsilon t} \tag{III.10}$$

Remark

The main point in the proof of the averaging theorem is the Gronwall's lemma, applied to an inequality like the following one (*cf.* Appendix I)

$$\begin{aligned} \|x-\xi\| &\le \varepsilon \int_0^t \|X(x,\tau)-X(\xi,\tau)\|d\tau + const \\ &\le \varepsilon\lambda_X \int_0^t \|x-\xi\|d\tau + const \end{aligned} \tag{III.11}$$

The last estimation is obtained due to the Lipschitz-continuity of X. This condition is not fulfilled for the right hand side of the equations in our case. But (III.11) shows that this condition should not be necessarily fulfilled for each time point, but it should be fulfilled in some integral sense. Thus before we prove the

Theorem, the following Lemma showing that the Lipschitz-condition in our case is fulfilled in the integral sense will be formulated and proved.

Lemma
Under the conditions of theorem the following inequality is fulfilled

$$\begin{aligned} N_1 = \Bigg\| \int_0^{\tau} \Big\{ Z(x_1,t)\Big[E\big(g(t)+\varepsilon f(x_1,t)\big)-E\big(g(t)\big)\Big]- \\ -Z(x_2,t)\Big[E\big(g(t)+\varepsilon f(x_2,t)\big)-E\big(g(t)\big)\Big]\Big\}\,dt\Bigg\| \le \\ \le C_{N1} \int_0^{(N+1)T} \|x_1-x_2\| \cdot \big|E\big(g(t)+\varepsilon K\big)-E\big(g(t)\big)\big|\,dt + \\ +C_{N2}\varepsilon \int_0^{(N+1)T} \|x_1-x_2\| \prod_{j=0}^{N}\prod_{i=1}^{m} \delta\big(t-(t_{i0}+jT)\big) \end{aligned} \tag{III.12}$$

Here $N=[\tau/T]$ is the integer part of τ/T, so that $NT \le \tau < (N+1)T$. δ is the standard Dirac's function.

Proof
First N_1 will be split into two parts using the triangle inequality:

$$\begin{aligned} N_1 \le \int_0^{\tau} \big\|Z(x_1,t)-Z(x_2,t)\big\| \cdot \big|E\big(g(t)+\varepsilon f(x_1,t)\big)-E\big(g(t)\big)\big|dt \\ + \int_0^{\tau} \big\|Z(x_2,t)\big\| \big|E\big(g(t)+\varepsilon f(x_1,t)\big)-E\big(g(t)+\varepsilon f(x_2,t)\big)\big|\,dt \end{aligned} \tag{III.13}$$

According to (III.7) and this estimation can be transformed as follows:

$$\begin{aligned} N_1 \le \lambda_Z \sum_{n=0}^{N} \int_{nT}^{(n+1)T} \|x_1-x_2\| \big|E\big(g(t)+\varepsilon f(x_1,t)\big)-E\big(g(t)\big)\big|dt \\ + M_Z \sum_{n=0}^{N} \int_{nT}^{(n+1)T} \big|E\big(g(t)+\varepsilon f(x_1,t)\big)-E\big(g(t)+\varepsilon f(x_2,t)\big)\big|\,dt \end{aligned} \tag{III.14}$$

Now the difference between two E-functions has to be estimated. But it is almost always equal to zero except in the vicinity of the zeros of $g(t)$. Consider the equation

$$g(t)+\varepsilon f(x,t)=0 \qquad \text{(III.15)}$$

Let us show, that this equation has at least one solution in the vicinity of $t_{i0}+jT$ (*cf.* (III.9)).

Assume $g'(t_{i0})<0$ and define:

$$t_{-}=t_{i0}-\varepsilon\frac{M_f}{G};\qquad t_{+}=t_{i0}+\varepsilon\frac{M_f}{G} \qquad \text{(III.16)}$$

The case $g'(t_{i0})>0$ can be investigated similarly.

The further analysis is illustrated in Fig. AIII.1.

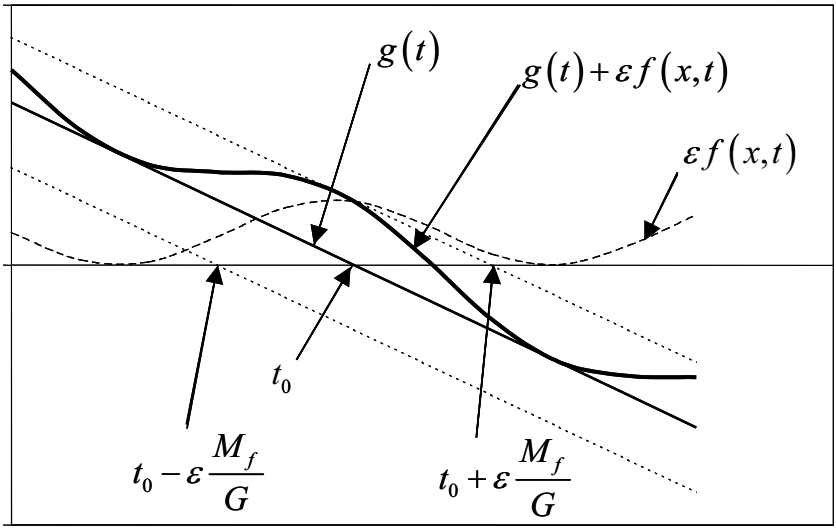

Fig. AIII.1. On the zeros of the function $g(t)+\varepsilon f(x,t)$

The basic idea is very simple: The function $f(x,t)$ is limited. Thus the function $g(t)+\varepsilon f(x,t)$ is bounded between $g(t)-\varepsilon M_f$ and $g(t)+\varepsilon M_f$. As a result the roots of $g(t)+\varepsilon f(x,t)$ are also bounded between the roots of the bounding functions. Hence $g(t_{-})$ can be estimated:

$$g(t_{-})=g(t_{-})-g(t_{i0})=\left|g(t_{-})-g(t_{i0})\right|\geq G\left|t_{-}-t_{i0}\right|=\varepsilon M_f \qquad \text{(III.17)}$$

Using the triangle inequality in the form $|a|-|b|\le|a-b|$ we obtain

$$\begin{aligned} g(t_-)+\varepsilon f(x,t_-) &\ge |g(t_-)|-\varepsilon|f(x,t_-)| \\ &\ge \varepsilon\left(M_f-|f(x,t_-)|\right)\ge 0 \end{aligned} \tag{III.18}$$

Similarly it can be shown that $g(t_{i0})-g(t_+)\ge\varepsilon M_f$ and

$$g(t_+)+\varepsilon f(x,t_+)\le -\varepsilon\left(M_f-|f(x,t_+)|\right)\le 0 \tag{III.19}$$

Hence between t_+ and t_- there is at least one switch point of the function

$$\left|E(g(t)+\varepsilon f(x,t))-E(g(t))\right| \tag{III.20}$$

And there are no switch points outside the interval until the next zero point of $g(t)$.

The maximal length of this interval can be estimated as follows:

$$|t_- - t_+|\le 2\varepsilon\frac{M_f}{G} \tag{III.21}$$

Now the difference (III.20) can be estimated in the vicinity of switch points:

$$\begin{aligned} &\left|E(g(t)+\varepsilon f(x,t))-E(g(t))\right| \\ &\qquad\le\left|E(g(t)+2\varepsilon M_f)-E(g(t))\right| \end{aligned} \tag{III.22}$$

This estimate does not depend on x, so it is correct uniformly with respect to x. Comparing (III.22) and (III.14) with (III.12) one obtains

$$C_{N1}=\lambda_Z;\qquad K=2M_f \tag{III.23}$$

Now the second term in (III.12) should be estimated. The solution of the equation (III.15) in the vicinity of t_i should be considered as a function of x.

Fig. AIII.2 illustrates the simple sense of the following analysis – if $g(t)+\varepsilon f(x,t)$ is a Lipschitz-continuous function with respect to x, then the solution of the equation $g(t)+\varepsilon f(x,t)=0$ is also a Lipschitz-continuous function:

$$t_i(x,\varepsilon):\ t_i(x,0)=t_{i0} \tag{III.24}$$

Equation (III.15) determines t_i as an implicit function of x and ε in the vicinity of $\varepsilon = 0$. In our conditions, according to the *Implicit-Function-Theorem* [98] (III.24) is a Lipschitz-continuous function with respect to x and ε. The corresponding Lipschitz-constant can be evaluated due to the supremum of the partial derivative of the function (III.24). It will be denoted as $\varepsilon\lambda_t$.

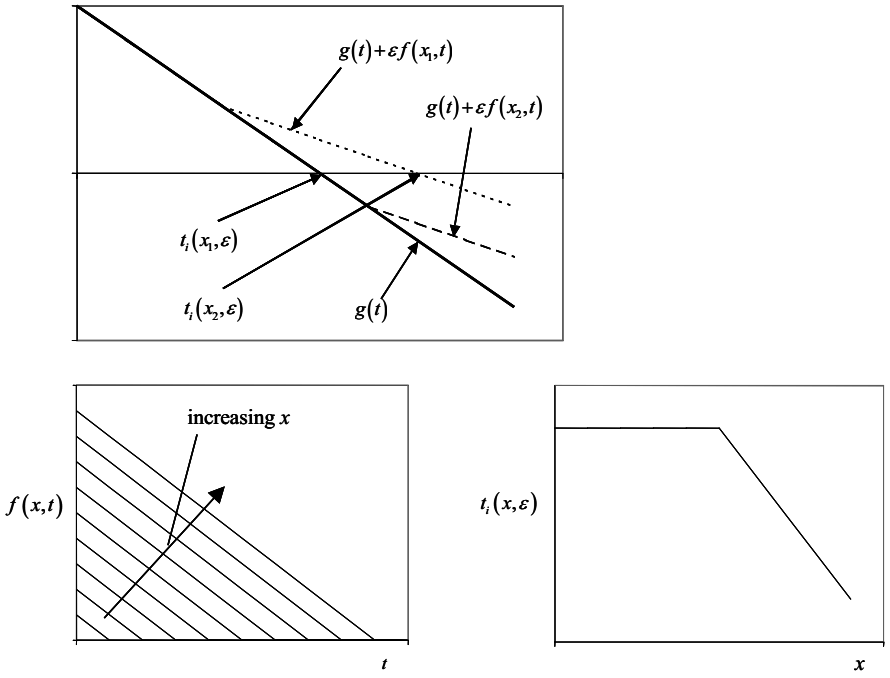

Fig. AIII.2. On the definition and Lipschitz-continuity of the implicit function $t_i(x,\varepsilon)$

Now the second term in (III.14) can be rewritten as follows:

$$\int_{nT}^{(n+1)T} \left| E\left(g(t)+\varepsilon f\left(x_1,t\right)\right) - E\left(g(t)+\varepsilon f\left(x_2,t\right)\right) \right| dt \le$$
$$\le \int_{\bigcup[t_i(x_1,\varepsilon)-t_i(x_2,\varepsilon)]} dt \le \varepsilon\lambda_t \int_{nT}^{(n+1)T} \left\| x_1 - x_2 \right\| \prod_{i=1}^{m} \delta\left(t - t_{i0} - nT\right) \tag{III.25}$$

Comparing the last estimation with (III.12) it can be seen that

$$C_{N2} = M_Z \lambda_t \tag{III.26}$$

The proof of the lemma is accomplished.

Now we can prove the theorem.

Proof of the theorem

The difference between the solutions to (III.1) and (III.3) has to be estimated. Because the initial conditions are the same, it is simple to see that

$$\begin{aligned} \|x-\xi\| \le \varepsilon & \left\| \int_0^t \left(X(x,\tau) - \Xi(\xi) \right) d\tau \right\| \\ + & \left\| \int_0^t \left\{ Z(x,\tau) \left[E\left(g(\tau) + \varepsilon f(x,\tau) \right) - E\left(g(\tau) \right) \right] - \varepsilon \varsigma(\xi) \right\} d\tau \right\| \end{aligned} \tag{III.27}$$

The estimation of these terms is now almost the same as for the proof of the standard averaging (Appendix I). The only difference is that the Lemma will be used here instead of the Lipschitz-condition in order to estimate the second term:

$$\begin{aligned} & \varepsilon \left\| \int_0^t \left(X(x,\tau) - \Xi(\xi) \right) d\tau \right\| \\ & + \left\| \int_0^t \left\{ Z(x,\tau) \left[E\left(g(\tau) + \varepsilon f(x,\tau) \right) - E\left(g(\tau) \right) \right] - \varepsilon \varsigma(\xi) \right\} d\tau \right\| \\ & \le \varepsilon \left\| \int_0^t \left(X(x,\tau) - X(\xi,\tau) \right) d\tau \right\| + \varepsilon \left\| \int_0^t \left(X(\xi,\tau) - \Xi(\xi) \right) d\tau \right\| \\ & + \left\| \int_0^t \left\{ Z(x,\tau) \left[E\left(g(\tau) + \varepsilon f(x,\tau) \right) - E\left(g(\tau) \right) \right] \right. \right. \\ & \left. \left. - Z(\xi,\tau) \left[E\left(g(\tau) + \varepsilon f(\xi,\tau) \right) - E\left(g(\tau) \right) \right] \right\} d\tau \right\| \\ & + \left\| \int_0^t \left\{ Z(\xi,\tau) \left[E\left(g(\tau) + \varepsilon f(\xi,\tau) \right) - E\left(g(\tau) \right) \right] - \varepsilon \varsigma(\xi) \right\} d\tau \right\| \end{aligned} \tag{III.28}$$

Now the Lipschitz-condition can be applied to the first term and the Lemma to the third term:

$$J_1 = \varepsilon \left\| \int_0^t \left(X(x,\tau) - X(\xi,\tau) \right) d\tau \right\| \le \varepsilon \lambda_X \int_0^{(N+1)T} \|x - \xi\| d\tau \tag{III.29}$$

$$\begin{aligned} J_3 = & \left\| \int_0^t \left\{ Z(x,\tau) \left[E\left(g(\tau) + \varepsilon f(x,\tau)\right) - E\left(g(\tau)\right) \right] \right.\right. \\ & \left.\left. - Z(\xi,\tau) \left[E\left(g(\tau) + \varepsilon f(\xi,\tau)\right) - E\left(g(\tau)\right) \right] \right\} d\tau \right\| \\ & \le C_{N1} \int_0^{(N+1)T} \|x - \xi\| \cdot \left| E\left(g(\tau) + \varepsilon K\right) - E\left(g(\tau)\right) \right| d\tau + \\ & + C_{N2} \varepsilon \int_0^{(N+1)T} \|x - \xi\| \cdot \prod_{j=0}^{N} \prod_{i=1}^{m} \delta\left(\tau - \left(t_{i0} + jT\right)\right) d\tau \end{aligned} \tag{III.30}$$

In order to estimate the second term $\xi_j = \xi(jT)$ can be introduced. Notice that the following identity is valid because Ξ is the average of X:

$$\left\| \int_{jT}^{(j+1)T} \left(X(\xi_j, t) - \Xi(\xi_j) \right) d\tau \right\| = 0 \tag{III.31}$$

Then

$$\begin{aligned} J_2 = & \left\| \varepsilon \int_0^{(N+1)T} \left(X(\xi,\tau) - \Xi(\xi) \right) d\tau \right\| \\ & \le \varepsilon \sum_{j=0}^{N} \left\| \int_{jT}^{(j+1)T} \left(X(\xi,\tau) - X(\xi_j,\tau) \right) d\tau \right\| \\ & + \varepsilon \sum_{j=0}^{N} \left\| \int_{jT}^{(j+1)T} \left(\Xi(\xi) - \Xi(\xi_j) \right) d\tau \right\| \\ & \le \varepsilon \sum_{j=0}^{N} 2\lambda_X \int_{jT}^{(j+1)T} \|\xi - \xi_j\| d\tau \end{aligned} \tag{III.32}$$

But ξ is the solution of the system

$$\dot{\xi} = \varepsilon \Xi(\xi) + \varepsilon \varsigma(\xi), \xi(jT) = \xi_j \tag{III.33}$$

Due to the conditions (III.7) and (III.8) it is easy to see, that

$$\begin{aligned} &\|\Xi\| \le M_X; \quad \|\varsigma\| \le 2mM_Z M_f / G \\ &\|\varsigma(x_1) - \varsigma(x_2)\| \le \lambda_\varsigma \|x_1 - x_2\|; \quad \lambda_\varsigma = m\left(2C_{N1} M_f / G + C_{N2}\right) \end{aligned} \tag{III.34}$$

Thus using (III.33) only for one period we find

$$\|\xi - \xi_j\| \le \varepsilon T \left(M_X + \frac{2mM_Z M_f}{G} \right) \tag{III.35}$$

Inserting (III.35) into (III.32) one obtains the estimation for the second term in (III.28):

$$J_2 \le 2\lambda_X \varepsilon^2 T^2 (N+1)\left(M_X + 2mM_Z M_f / G\right) \tag{III.36}$$

The last term in (III.28) can be estimated similarly but the Lipschitz-condition in (III.32) must be replaced by the Lemma:

$$\begin{aligned} J_4 &= \left\| \int_0^{(N+1)T} \left\{ Z(\xi,\tau)\Delta E(\xi,\tau,\varepsilon) - \varepsilon\varsigma(\xi) \right\} d\tau \right\| \\ &\le \varepsilon T \left(M_X + \frac{2mM_Z M_f}{G} \right) \times \\ &\sum_{j=0}^{N} \int_{jT}^{(j+1)T} \left\{ \lambda_Z \|\Delta E(\xi,\tau,\varepsilon)\| + \varepsilon\lambda_t \prod_{i=1}^{m} \delta\left(\tau - (t_{i0} + jT)\right) + \varepsilon\lambda_\varsigma \right\} d\tau \\ &\Delta E(\xi,\tau,\varepsilon) \equiv E\left(g(\tau) + \varepsilon f(\xi,\tau)\right) - E\left(g(\tau)\right) \end{aligned} \tag{III.37}$$

The last integral can be easily estimated, because (see (III.21))

$$\begin{aligned} &\sum_{j=0}^{N} \int_{jT}^{(j+1)T} \|\Delta E(\xi,\tau,\varepsilon)\| d\tau \le 2(N+1) m\varepsilon \frac{M_f}{G} \\ &\sum_{j=0}^{N} \int_{jT}^{(j+1)T} \prod_{i=1}^{m} \delta\left(\tau - (t_{i0} + jT)\right) d\tau = (N+1)m \end{aligned} \tag{III.38}$$

Hence

$$J_4 \le \varepsilon^2 T\left(N+1\right)\left(M_X + 2mM_Z M_f / G\right) \times\left(2m\lambda_Z M_f / G + m\lambda_t + T\lambda_\varsigma\right) \tag{III.39}$$

The last step is to apply the Gronwall's lemma to (III.28). It gives us the final estimate:

$$\left\|x-\xi\right\| \le \left(J_2 + J_4\right) e^{\varepsilon\left(N+1\right)\left(\lambda_X T + 2m\lambda_Z M_f / G + m\lambda_t\right)} \tag{III.40}$$

This inequality gives the required accuracy and fulfils the proof of the Theorem.

Appendix IV: Averaging of Systems with Small Discontinuities of the Right Hand Sides

Theorem

Consider the initial value problem

$$\begin{aligned} \dot{x} &= \varepsilon Z\left(x,t\right) E\left(g\left(t\right)+f\left(x\right)\right) \\ x\left(0\right) &= x_0 \end{aligned} \tag{IV.1}$$

Here $E\left(t\right)$ is the "one step" function (*cf.* Appendix III).

Consider the corresponding averaged problem alongside with (IV.1):

$$\begin{aligned} \dot{\xi} &= \varepsilon\varsigma\left(\xi\right), \xi\left(0\right) = x_0 \\ \varsigma\left(\xi\right) &= \left\langle Z\left(\xi,t\right) E\left(g\left(t\right)+f\left(\xi\right)\right)\right\rangle_t \end{aligned} \tag{IV.2}$$

Here

$$\begin{aligned} &Z : R^{n+1} \to R^n \\ &f : R^n \to W_f \subset R^1 \\ &g : R^1 \to W_g \subset R^1, \qquad W_f \subset W_g \\ &x, \xi, x_0 \in D \subset R^n, t \in \left[0,\infty\right), \varepsilon \in \left(0, \varepsilon_0\right] \end{aligned} \tag{IV.3}$$

Suppose

1. Z and f are measurable functions of t for constant x and ε.
2. All the functions are T-periodic with respect to t. $\left\langle \ \right\rangle_t$ means the average with respect to time.
3. Z is a bounded Lipschitz-continuous function in x on D, i.e.

$$\left\|Z(x,t)\right\| \le M_Z; \quad \left\|Z(x_1,t)-Z(x_2,t)\right\| \le \lambda_Z \left\|x_1-x_2\right\| \tag{IV.4}$$

4. f is a bounded differentiable function in x on D with Lipschitz-constant λ_f, i.e.

$$\left\|f(x,t)\right\| \le M_f; \quad \left\|f(x_1,t)-f(x_2,t)\right\| \le \lambda_f \left\|x_1-x_2\right\| \tag{IV.5}$$

5. $g(t) \in C^{(1)}\left[0,\infty\right)$

6. The equation $g(t) = const \in W_f$ has m solutions $t_{i0}, i = 1, \ldots, m$ for $t \in [0, T)$ and

$$\left| g'(t_{i0}) \right| \geq G > 0 \tag{IV.6}$$

This condition is illustrated in Fig. AIV.1.

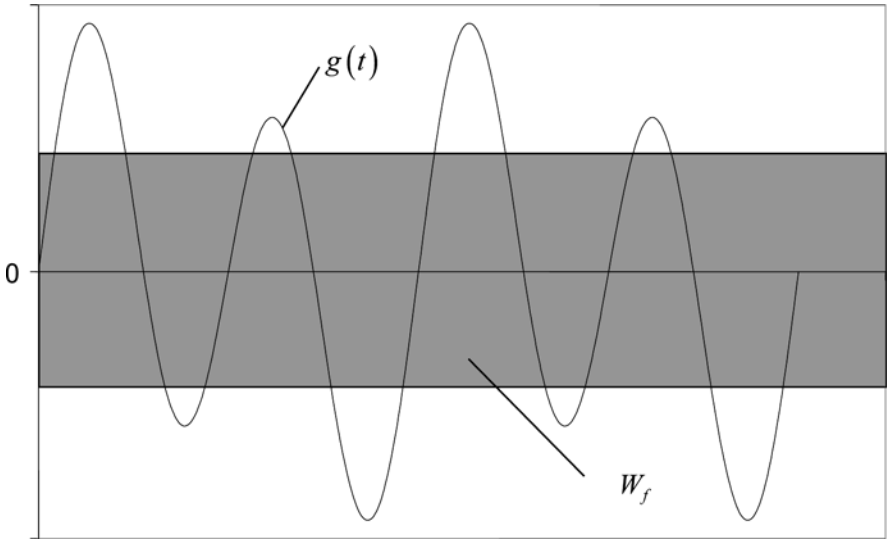

Fig. AIV.3. The derivative of the function is not equal to zero in the whole domain W_f

7. All the constants does not depend on ε; ξ belongs to the interior subset of D on the time scale $1/\varepsilon$.

Then the solutions to (IV.1) and (IV.2) are asymptotically close to each other, i.e. the error one makes on using the averaged system instead of the original one is small for the asymptotically long time interval:

$$\left\| x(t) - \xi(t) \right\| \leq C_1 \varepsilon e^{C_2 \varepsilon t} \tag{IV.7}$$

Proof

The proof is very similar to that in Appendix III. The only difference is the necessity to estimate the term

$$N_3 = \varepsilon \int_0^{(N+1)T} \left| E\left(g(t)+f(x_1)\right) - E\left(g(t)+f(x_2)\right) \right| \, dt$$
$$\leq \varepsilon \sum_{n=0}^{N} \int_{nT}^{(n+1)T} \left| E\left(g(t)+f(x_1)\right) - E\left(g(t)+f(x_2)\right) \right| \, dt \tag{IV.8}$$

According to (IV.1) and (IV.5) the following inequality is valid:

$$\left|\frac{df}{dt}\right| = \left|\frac{df}{dx}\dot{x}\right| = \varepsilon \left|\frac{df}{dx} Z(x,t) E\left(g(t)+f(x)\right)\right| \leq \varepsilon \lambda_f M_Z \tag{IV.9}$$

It follows from the condition (IV.6) and the theorem on the existence and differentiability of the implicit function [98] that the following equation has in each interval $t \in \left[nT,(n+1)T\right)$ the m differentiable solutions satisfying the Lipschitz-condition and the corresponding constant is:

$$g(t)+f(x)=0 \rightarrow t_i = t_i(x),\ \lambda_t = \frac{\lambda_f}{G} \tag{IV.10}$$

Applying (IV.9) and (IV.10) to (IV.8) we obtain

$$N_3 \leq \varepsilon \sum_{n=0}^{N} \int_{nT}^{(n+1)T} \left| E\left(g(t)+f(x_1)\right) - E\left(g(t)+f(x_2)\right) \right| dt$$
$$\leq \varepsilon \sum_{n=0}^{N} \sum_{i=1}^{m} \left| t_i(x_1) - t_i(x_2) \right| \leq \varepsilon m \frac{\lambda_f}{G} \left\| x_1 - x_2 \right\| \tag{IV.11}$$

The remaining proof is trivial:

$$\begin{aligned}
\|x-\xi\| &\le \varepsilon \left\| \int_0^t \left\{ Z(x,\tau) E\left(g(\tau)+f(x)\right) - \varepsilon\varsigma(\xi) \right\} d\tau \right\| \\
&\le \varepsilon \left\| \int_0^t \left\{ Z(x,\tau) - Z(\xi,\tau) \right\} E\left(g(\tau)+f(x)\right) d\tau \right\| \\
&+ \varepsilon \left\| \int_0^t Z(x,\tau) \left\{ E\left(g(\tau)+f(x)\right) - E\left(g(\tau)+f(\xi)\right) \right\} d\tau \right\| \\
&+ \varepsilon \left\| \int_0^t \left\{ Z(\xi,\tau) E\left(g(\tau)+f(\xi)\right) - \varepsilon\varsigma(\xi) \right\} d\tau \right\|
\end{aligned} \tag{IV.12}$$

The first term here can be estimated according to (IV.4). The relationship (IV.11) can be applied to the second term. The last one can be estimated exactly like in the standard averaging. Finally the Gronwall's lemma accomplishes the proof.

Appendix V: Averaging of Systems with Small Discontinuities of the Unknown Function

Lemma (the Gronwall's lemma for sequences)
Consider the sequence of inequalities

$$w_k \le \sum_{i=0}^{k} a_i w_i + b,\ k = 1,2,\ldots \tag{V.1}$$

Suppose

$$w_k > 0,\ 1 > a_k > 0,\ b > 0,\ k = 1,2,\ldots \tag{V.2}$$

Then the following inequalities are also fulfilled:

$$w_k \le \frac{b}{\prod_{i=0}^{k}(1-a_i)} \tag{V.3}$$

Proof

Let us introduce $z_k = w_k a_k$ and rewrite the inequalities (V.1) as follows

$$z_k \le a_k \sum_{i=0}^{k} z_i + b a_k \tag{V.4}$$

The next step of the proof can be done by induction. Consider $n = 1$:

$$z_0 \le a_0 z_0 + b a_0 \Rightarrow z_0 \le \frac{b a_0}{(1-a_0)} \tag{V.5}$$

For $n = 2$ we obtain

$$\begin{aligned} &z_1 \le a_1 (z_0 + z_1) + b a_1 \Rightarrow \\ &z_1 \le \frac{a_1 z_0 + b a_1}{1-a_1} \le \frac{b a_1}{(1-a_0)(1-a_1)} = \frac{b a_1}{\prod_{i=0}^{1}(1-a_i)} \end{aligned} \tag{V.6}$$

Suppose for certain k the following inequality is fulfilled:

$$z_k \le \frac{ba_k}{\prod_{i=0}^{k}(1-a_i)} \tag{V.7}$$

Consider the next element z_{k+1}:

$$z_{k+1} \le a_{k+1}b + a_{k+1}\sum_{i=0}^{k+1} z_i = a_{k+1}b + a_{k+1}\sum_{i=0}^{k} z_i + a_{k+1}z_{k+1}$$

$$\Rightarrow z_{k+1} \le \frac{a_{k+1}}{(1-a_{k+1})}\left(b + \sum_{i=0}^{k} z_i\right) \le \frac{a_{k+1}b}{(1-a_{k+1})}\left(1 + \sum_{i=0}^{k} \frac{a_k}{\prod_{j=0}^{i}(1-a_j)}\right) \tag{V.8}$$

Consider the term in brackets:

$$\prod_{i=0}^{k}(1-a_i) + \sum_{i=0}^{k} a_i \prod_{j=i+1}^{k}(1-a_j) = \prod_{i=1}^{k}(1-a_i) - a_0\prod_{i=1}^{k}(1-a_i) + a_0\prod_{i=1}^{k}(1-a_i) + \sum_{i=1}^{k} a_i \prod_{j=i+1}^{k}(1-a_j) = 1 \tag{V.9}$$

Hence

$$z_{k+1} \le \frac{a_{k+1}b}{(1-a_{k+1})\prod_{j=0}^{k}(1-a_j)} = \frac{a_{k+1}b}{\prod_{j=0}^{k+1}(1-a_j)} \tag{V.10}$$

We have proved the relationship (V.7) for $k=1$. Thus it is valid for an arbitrary k. Returning back to the variables w_k we obtain the required inequality.

Remark (Relationship to the classical Gronwall's lemma)
Consider the case

$$a_k = \frac{a}{n},\ n >> 1,\ k = 1,2,\ldots \tag{V.11}$$

Then the inequality (V.3) can be transformed as follows for large numbers $k = n$:

$$w_n \le \frac{b}{\left(1-\frac{a}{n}\right)^n} \tag{V.12}$$

Now the classical estimate can be applied

$$\left(1+ax\right)^{\frac{1}{x}} \le e^a \tag{V.13}$$

We use the transformation

$$\frac{1}{1-\frac{a}{n}} = 1+ax \Rightarrow n = \frac{1}{x}+a \tag{V.14}$$

Then the following estimate can be obtained:

$$w_n \le \frac{b}{\left(1-\frac{a}{n}\right)^n} = b\left(1+ax\right)^{\frac{1}{x}+a} \le be^a\left(1+ax\right)^a = \frac{be^a}{\left(1-\frac{a}{n}\right)^a} \tag{V.15}$$

Now we can formulate and prove the following Theorem.

Theorem
Consider a system

$$\begin{aligned} &\frac{dx}{dt} = \varepsilon X\left(x,t,\varepsilon\right) \text{ if } t \ne 2\pi n \\ &\Delta x \equiv x_+ - x_- = \varepsilon f\left(x_-\right) \text{ if } t = 2\pi n \\ &n = 1,2,\ldots \end{aligned} \tag{V.16}$$

Here $x \in D \subset R^n$, x_- and x_+ are the values of the phase variables before and after the passage of the independent variable t through the value $2\pi n$.

Suppose

1. X is a measurable function of t for constant x and ε.
2. Function X is 2π-periodic with respect to t. $\left\langle \ \right\rangle_t$ means the average.
3. X is a bounded Lipschitz-continuous function in x on D, i.e.

$$\begin{aligned} &\left\|X\left(x,t,\varepsilon\right)\right\| \le M_X \\ &\left\|X\left(x_1,t,\varepsilon\right)-X\left(x_2,t,\varepsilon\right)\right\| \le \lambda_X \left\|x_1-x_2\right\| \end{aligned} \tag{V.17}$$

4. $f\left(x\right)$ is bounded and Lipschitz-continuous in x on D

$$\begin{aligned} &\left\|f\left(x\right)\right\| \le M_f \\ &\left\|f\left(x_1\right)-f\left(x_2\right)\right\| \le \lambda_f \left\|x_1-x_2\right\| \end{aligned} \tag{V.18}$$

5. All constants do not depend on ε.
6. Consider the averaged system alongside the original one.

$$\begin{aligned} &\dot{\xi} = \varepsilon\Xi\left(\xi,\varepsilon\right)+\frac{\varepsilon}{2\pi}f\left(\xi\right), \qquad \xi\left(0\right)=x_0 \\ &\Xi\left(\xi,\varepsilon\right)=\left\langle X\left(\xi,t,\varepsilon\right)\right\rangle_t \end{aligned} \tag{V.19}$$

Suppose its solution belongs to the interior subset of D on the t scale $1/\varepsilon$. Under these conditions solutions of the initial value problems (V.16) and (V.19) are asymptotically close to each other, i.e.

$$\left\|x-\xi\right\| \le C_1\varepsilon e^{\varepsilon C_2 t} \tag{V.20}$$

Proof

The proof is similar to that in Appendix I. (We omit the dependence of X on ε in order to shorten the notation.)

We estimate the difference between the solutions to (V.16) and (V.19):

$$\begin{aligned} \left\|x-\xi\right\| &\le \varepsilon\left\|\int_0^t\left(X\left(x,\tau\right)-\Xi\left(\xi\right)\right)d\tau\right\| \\ &\quad +\varepsilon\left\|\sum_{i=0}^{N}f\left(x_{\_}\left(\pi i\right)\right)-\frac{1}{2\pi}\int_0^t f\left(\xi\right)d\tau\right\| \\ &\le \varepsilon\int_0^t\left\|X\left(x,\tau\right)-X\left(\xi,\tau\right)\right\|d\tau+\varepsilon\sum_{i=0}^{N}\left\|f\left(x_{\_}\left(\pi i\right)\right)-f\left(\xi\left(\pi i\right)\right)\right\| \\ &\quad +\varepsilon\int_0^t\left\|X\left(\xi,\tau\right)-\Xi\left(\xi\right)\right\|d\tau+\varepsilon\left\|\sum_{i=0}^{N}f\left(\xi\left(\pi i\right)\right)-\frac{1}{2\pi}\int_0^t f\left(\xi\right)d\tau\right\| \end{aligned} \tag{V.21}$$

Here $N=\left[t/2\pi\right]$ is the number of discontinuous jumps in the considered time interval.

The first two terms in (V.21) can be estimated according to the assumptions (V.17) and (V.18). The time interval can be increased to an integer number of periods:

$$\begin{aligned}\left\|x-\xi\right\| &\le \varepsilon\lambda_X \int_0^t \left\|x-\xi\right\| d\tau + \varepsilon \int_0^{2\pi(N+1)} \left\|X\left(\xi,\tau\right)-\Xi\left(\xi\right)\right\| d\tau \\ &+\varepsilon\lambda_f \sum_{i=0}^{N} \left\|x_{-}\left(2\pi i\right)-\xi\left(2\pi i\right)\right\| \\ &+\varepsilon\left\|\sum_{i=0}^{N} f\left(\xi\left(2\pi i\right)\right)-\frac{1}{2\pi}\int_0^{2\pi(N+1)} f\left(\xi\right)d\tau\right\|\end{aligned} \tag{V.22}$$

Let us denote $\xi\left(2\pi i\right)=\xi_i$. Notice that according to the definition of Ξ the following relationship is valid:

$$\begin{aligned}&\left\|\int_{2\pi i}^{2\pi(i+1)} \left(X\left(\xi_i,\tau\right)-\Xi\left(\xi_i\right)\right)d\tau\right\|=0; \\ &\left\|f\left(\xi_i\right)-\frac{1}{2\pi}\int_{2\pi i}^{2\pi(i+1)} f\left(\xi_i\right)d\tau\right\|=0;\quad i=1,2,3,\ldots\end{aligned} \tag{V.23}$$

The second term in (V.22) can be estimated exactly as in Appendix I:

$$\begin{aligned}&\int_0^{(N+1)\pi} \left\|X\left(\xi,\tau\right)-\Xi\left(\xi\right)\right\| d\tau \\ &\le \sum_{i=0}^{N}\left\|\int_{2\pi i}^{2\pi(i+1)} \left(X\left(\xi,\tau\right)-X\left(\xi_i,\tau\right)\right)d\tau\right\| \\ &\qquad +\sum_{i=0}^{N}\left\|\int_{2\pi i}^{2\pi(i+1)} \left(\Xi\left(\xi\right)-\Xi\left(\xi_i\right)\right)d\tau\right\| \\ &\le 2\lambda_X \sum_{i=0}^{N}\int_{2\pi i}^{2\pi(i+1)} \left\|\xi-\xi_i\right\| d\tau\end{aligned} \tag{V.24}$$

Let us estimate the last term in (V.22):

$$\left\|\sum_{i=0}^{N} f\left(\xi(2\pi i)\right)-\frac{1}{2\pi}\int_{0}^{2\pi(N+1)} f(\xi)\,d\tau\right\| \\ \le \frac{1}{2\pi}\sum_{i=0}^{N}\int_{2\pi i}^{2\pi(i+1)}\left\|f(\xi_i)-f(\xi)\right\|d\tau \le \frac{\lambda_f}{2\pi}\sum_{i=0}^{N}\int_{2\pi i}^{2\pi(i+1)}\left\|\xi_i-\xi\right\|d\tau \tag{V.25}$$

Inserting (V.24) and (V.25) into (V.22) we obtain

$$\begin{aligned}\left\|x-\xi\right\| \le{}& \varepsilon\lambda_X\int_0^t\left\|x-\xi\right\|d\tau+\varepsilon\lambda_f\sum_{i=0}^{N}\left\|x_-(2\pi i)-\xi_i\right\| \\ &+\varepsilon\left(2\lambda_X+\frac{\lambda_f}{2\pi}\right)\sum_{i=0}^{N}\int_{2\pi i}^{2\pi(i+1)}\left\|\xi-\xi_i\right\|d\tau\end{aligned} \tag{V.26}$$

Integrating the equations (V.19) for one period we obtain:

$$\begin{aligned}&\dot{\xi}=\varepsilon\Xi(\xi,\varepsilon)+\frac{\varepsilon}{2\pi}f(\xi),\ \xi(2\pi i)=\xi_i \\ &\Rightarrow\left\|\xi-\xi_i\right\|\le\varepsilon\left(2\pi M_X+M_f\right)\end{aligned} \tag{V.27}$$

Thus the estimation (V.26) can be transformed as follows:

$$\begin{aligned}\left\|x-\xi\right\| \le{}& \varepsilon\lambda_X\int_0^t\left\|x-\xi\right\|d\tau+\varepsilon\lambda_f\sum_{i=0}^{N}\left\|x_-(2\pi i)-\xi_i\right\| \\ &+\varepsilon^2\left(4\pi\lambda_X+\lambda_f\right)\left(2\pi M_X+M_f\right)(N+1)\end{aligned} \tag{V.28}$$

In particular we can estimate the difference $w_k=\left\|x_-(2\pi k)-\xi_k\right\|$:

$$\begin{aligned}w_k \le{}& \varepsilon\lambda_f\sum_{i=0}^{k}w_i+\varepsilon\lambda_X\int_0^t\left\|x-\xi\right\|d\tau \\ &+\varepsilon^2\left(4\pi\lambda_X+\lambda_f\right)\left(2\pi M_X+M_f\right)(N+1)\end{aligned} \tag{V.29}$$

Applying the Gronwall's lemma for sequences we obtain:

$$w_k \le \frac{\varepsilon\lambda_X}{\left(1-\varepsilon\lambda_f\right)^k}\int_0^t \left\|x-\xi\right\| d\tau + \frac{\varepsilon^2}{\left(1-\varepsilon\lambda_f\right)^k}\left(4\pi\lambda_X+\lambda_f\right)\left(2\pi M_X+M_f\right)(N+1) \tag{V.30}$$

We must also estimate the following term:

$$\sum_{i=0}^{N}\frac{1}{\left(1-\varepsilon\lambda_f\right)^i} \le \sum_{i=0}^{N}\frac{1}{\left(1-\varepsilon\lambda_f\right)^N} = \frac{N+1}{\left(1-\varepsilon\lambda_f\right)^N} \tag{V.31}$$

Substituting this inequality into (V.28) we get:

$$\left\|x-\xi\right\| \le \varepsilon\lambda_X\left(1+\frac{\varepsilon\lambda_f(N+1)}{\left(1-\varepsilon\lambda_f\right)^N}\right)\int_0^t\left\|x-\xi\right\|d\tau + \varepsilon^2\left(4\pi\lambda_X+\lambda_f\right)\left(2\pi M_X+M_f\right)(N+1)\left(1+\frac{\varepsilon\lambda_f(N+1)}{\left(1-\varepsilon\lambda_f\right)^N}\right) \tag{V.32}$$

Now the classical Gronwall's lemma can be applied:

$$\left\|x-\xi\right\| \le \varepsilon C_1 e^{\varepsilon C_2 t}$$

$$C_1 = \varepsilon\left(4\pi\lambda_X+\lambda_f\right)\left(2\pi M_X+M_f\right)(N+1)\left(1+\frac{\varepsilon\lambda_f(N+1)}{\left(1-\varepsilon\lambda_f\right)^N}\right) \tag{V.33}$$

$$C_2 = \lambda_X\left(1+\frac{\varepsilon\lambda_f(N+1)}{\left(1-\varepsilon\lambda_f\right)^N}\right)$$

This estimation accomplishes the proof of the theorem. In particular for $\varepsilon = \varepsilon_0/N,\ \varepsilon_0 = O(1)$ we can apply (V.15) and obtain the last estimate:

$$C_1 = \left(4\pi\lambda_X+\lambda_f\right)\left(2\pi M_X+M_f\right)\left(1+\lambda_f\right)+O(\varepsilon)$$
$$C_2 = \lambda_X\left(1+\lambda_f\right)+O(\varepsilon) \tag{V.34}$$

Appendix VI: Averaging of Variable Order Discontinuous Systems

Theorem 1

(The following theorem and its proof are based on [33].)

Consider a system described over certain times by differential equations, and at other time intervals by differential and finite relations of the following form:

$$\begin{aligned} &\dot{x} = \varepsilon X\left(x, yM\left(t\right), t, \varepsilon\right) \\ &\dot{y}M\left(t\right) = \varepsilon Y\left(x, yM\left(t\right), t, \varepsilon\right) M\left(t\right) \\ &y\left(2\pi n\right) = G\left(x\left(2\pi n\right), \varepsilon\right), \quad n = 0,1,2, \ \ldots \\ &x\left(0\right) = x_0 \end{aligned} \tag{VI.1}$$

Here $M\left(t\right)$ is a 2π-periodic piecewise constant function:

$$M\left(t\right) = \begin{cases} 1, & 0 \le t < \pi \\ 0, & \pi \le t < 2\pi \end{cases}, \quad M\left(t + 2\pi\right) = M\left(t\right) \tag{VI.2}$$

Consider the averaged system alongside with (VI.1):

$$\begin{aligned} &\dot{\xi} = \varepsilon \Xi\left(\xi, \eta, \varepsilon\right) \equiv \varepsilon \left\langle X\left(\xi, \eta M\left(t\right), t, \varepsilon\right)\right\rangle_t \\ &\eta = G\left(\xi, \varepsilon\right) \\ &\xi\left(0\right) = x_0 \end{aligned} \tag{VI.3}$$

Here $X:\ R^{k+l+2} \to R^k; Y:\ R^{k+l+2} \to R^l;\ G:\ R^{k+1} \to D_y \subset R^l$ and $x, x_0, \xi \in D_x \subset R^k;\ \ y, \eta \in D_y \subset R^l;\ \ t \in \left[0, \infty\right);\ \ \varepsilon \in \left(0, \varepsilon_0\right].$

Notice that the vector function $y\left(t\right)$ is a solution of an infinite sequence of systems of differential equation. Each of them is valid together with the continuous equations for x in the time intervals $2\pi n \le t < \left(2n+1\right)\pi$. In the time intervals $\left(2n+1\right)\pi \le t < \left(2n+2\right)\pi$ there are only equations for x. At the end of each of these intervals initial conditions for y are reset.

Suppose:

1. X and Y are measurable functions of t for constant x, y and ε.
2. All the functions are 2π -periodic with respect to t. $\left\langle\ \right\rangle_t$ means the time average:

$$\left\langle f(t)\right\rangle_t = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt \tag{VI.4}$$

3. X and Y are bounded Lipschitz-continuous functions in x and y on $D_x \cup D_y$, i.e.:

$$\begin{aligned}
&\left\|X(x,yM(t),t,\varepsilon)\right\| \le M_X; \quad \left\|Y(x,yM(t),t,\varepsilon)\right\| \le M_Y \\
&\left\|X(x_1,y_1M(t),t,\varepsilon) - X(x_2,y_2M(t),t,\varepsilon)\right\| \\
&\qquad \le \lambda_{XX}\left\|x_1-x_2\right\| + \lambda_{XY}\left\|y_1-y_2\right\|M(t) \\
&\left\|Y(x_1,y_1M(t),t,\varepsilon) - Y(x_2,y_2M(t),t,\varepsilon)\right\| \\
&\qquad \le \lambda_{YX}\left\|x_1-x_2\right\| + \lambda_{YY}\left\|y_1-y_2\right\|M(t)
\end{aligned} \tag{VI.5}$$

4. G is a bounded function, satisfying Lipschitz-condition in x on D_x together with its first partial derivatives with respect to x, i.e.

$$\begin{aligned}
&\left\|G(x,t)\right\| \le M_G \\
&\left\|G(x_1,t) - G(x_2,t)\right\| \le \lambda_G\left\|x_1-x_2\right\| \\
&\left\|\left.\frac{\partial G(x,t)}{\partial x}\right|_{x_1} - \left.\frac{\partial G(x,t)}{\partial x}\right|_{x_2}\right\| \le \lambda_{G1}\left\|x_1-x_2\right\|
\end{aligned} \tag{VI.6}$$

5. All constants do not depend on ε, and ξ belongs to the interior subset of D_x on the time scale $1/\varepsilon$.

Then the solutions to (VI.1) and (VI.3) are asymptotically close to each other, i.e. the error one makes if using the averaged system instead of the original one is small for the asymptotically long time interval:

$$\begin{aligned}
&\left\|x-\xi\right\| \le C_1\varepsilon^2 te^{C_2\varepsilon t} \\
&\left\|yM(t)-\eta M(t)\right\| \le C_3\varepsilon^2 te^{C_4\varepsilon t}M(t)
\end{aligned} \tag{VI.7}$$

Remark

Here and further we use the alternative approach for proving the averaging, which is much closer to the basic idea of almost identical transformations.

Proof

The main idea of the proof is to redefine system (VI.1) in the domain where $M(t)=0$:

$$\begin{aligned}
&\dot{x}_1 = \varepsilon X\left(x_1, y_1 M(t), t, \varepsilon\right)\\
&\dot{y}_1 = \varepsilon Y\left(x_1, y_1 M(t), t, \varepsilon\right) + 2\varepsilon H\left(x_1, y_1, \varepsilon\right)\left(1 - M(t)\right)\\
&y_1(2\pi n) = G\left(x_1(2\pi n), \varepsilon\right), \quad x_1(0) = x_0
\end{aligned} \tag{VI.8}$$

Here $H(x_1, y_1, \varepsilon)$ is an arbitrary bounded function satisfying the Lipschitz-condition. It will be specified later. Due to the uniqueness of the solution of the initial value problem, the solutions to (VI.1) and to (VI.8) are identical in the following sense:

$$x_1 \equiv x; \;\; y_1 M(t) \equiv y M(t) \tag{VI.9}$$

Now the standard averaging procedure can be applied to the system (VI.8). It means the almost identical change of variables

$$\begin{aligned}
x_1 &= \xi_1 + \varepsilon u\left(\xi_1, \eta_1, t, \varepsilon\right)\\
y_1 &= \eta_1 + \varepsilon v\left(\xi_1, \eta_1, t, \varepsilon\right)
\end{aligned} \tag{VI.10}$$

Here u and v are periodic functions of t, their average is zero. The conditions for standard averaging are fulfilled here and the functions u and v can be chosen as follows:

$$\begin{aligned}
\Xi &= \left\langle X\left(\xi_1, \eta_1 M(t), t, \varepsilon\right)\right\rangle_t\\
\Psi &= \left\langle Y\left(\xi_1, \eta_1 M(t), t, \varepsilon\right)\right\rangle_t\\
u &= \int_0^t \left\{X\left(\xi_1, \eta_1 M(\tau), \tau, \varepsilon\right) - \Xi\right\} d\tau\\
v &= \int_0^t \left\{Y\left(\xi_1, \eta_1 M(\tau), \tau, \varepsilon\right) - \Psi\right\} d\tau + H\int_0^t \left\{1 - 2M(\tau)\right\} d\tau
\end{aligned} \tag{VI.11}$$

The integration here is performed with respect to the explicitly appearing time and takes $M(t)$ into account.

The arbitrariness of the function H can be used now. We choose it as follows:

$$H\left(\xi_1,\eta_1,\varepsilon\right)=\frac{\partial G\left(\xi_1,\varepsilon\right)}{\partial\xi_1}\Xi-\Psi \tag{VI.12}$$

The new variables ξ_1,η_1 are governed by the equations

$$\begin{aligned}\dot{\xi}_1&=\varepsilon\Xi+\varepsilon^2R_1\\ \dot{\eta}_1&=\frac{\partial G}{\partial\xi_1}\Xi+\varepsilon^2R_2\\ \eta_1\left(2\pi n\right)&=G\left(\xi_1\left(2\pi n\right),\varepsilon\right)+\varepsilon R_3\end{aligned} \tag{VI.13}$$

Here R_1,R_2 and R_3 are the residuals. In order to estimate them let us firstly estimate the following terms:

$$\begin{aligned}&\left\|\Xi\right\|\le M_X;\ \left\|\Psi\right\|\le M_Y\\ &\left\|\Xi\left(\xi_1,\eta_1,\varepsilon\right)-\Xi\left(\xi_2,\eta_2,\varepsilon\right)\right\|\le\lambda_{XX}\left\|\xi_1-\xi_2\right\|+\lambda_{XY}\left\|\eta_1-\eta_2\right\|\\ &\left\|H\right\|\le\lambda_G M_X+M_Y\\ &\left\|H\left(\xi_1,\eta_1,\varepsilon\right)-H\left(\xi_2,\eta_2,\varepsilon\right)\right\|\le\lambda_{HX}\left\|\xi_1-\xi_2\right\|+\lambda_{HY}\left\|\eta_1-\eta_2\right\|\\ &\lambda_{HX}=\lambda_{YX}+\lambda_G\lambda_{XX}+\lambda_{G1}M_X;\ \lambda_{HY}=\lambda_{YY}+\lambda_G\lambda_{XY}\\ &\left\|u\right\|\le 2\pi M_X;\ \left\|v\right\|\le 2\pi\left(M_Y+\left\|H\right\|\right)\le 2\pi\left(2M_Y+\lambda_G M_X\right)\\ &\left\|\frac{\partial u}{\partial\xi_1}\right\|\le 2\pi\lambda_{XX};\ \left\|\frac{\partial u}{\partial\eta_1}\right\|\le 2\pi\lambda_{XY}\\ &\left\|\frac{\partial v}{\partial\xi_1}\right\|\le 2\pi\left(\lambda_{YX}+\lambda_{HX}\right)\le 2\pi\left(2\lambda_{YX}+\lambda_G\lambda_{XX}+\lambda_{G1}M_X\right)\\ &\left\|\frac{\partial v}{\partial\eta_1}\right\|\le 2\pi\left(\lambda_{YY}+\lambda_{HY}\right)\le 2\pi\left(2\lambda_{YY}+\lambda_G\lambda_{XY}\right)\end{aligned} \tag{VI.14}$$

Now we can easily estimate the residuals. The first one is:

$$\begin{aligned}\left\|R_1\right\|&\le\lambda_{XX}\left\|u\right\|+\lambda_{XY}\left\|v\right\|+\left\|\frac{\partial u}{\partial\xi_1}\right\|\left\|\Xi\right\|+\left\|\frac{\partial u}{\partial\eta_1}\right\|\left\|\Psi\right\|\\ &\le 4\pi M_X\lambda_{XX}+4\pi\left(2M_Y+\lambda_G M_X\right)\lambda_{XY}\end{aligned} \tag{VI.15}$$

The following estimation is valid for the second one:

$$\begin{aligned}\|R_2\| &\le (\lambda_{YX}+\lambda_{HX})\|u\|+(\lambda_{YY}+\lambda_{HY})\|v\| \\ &\quad +\left\|\frac{\partial v}{\partial \xi_1}\right\|\|\Xi\|+\left\|\frac{\partial v}{\partial \eta_1}\right\|\|\Psi\| \\ &\le 4\pi M_X(\lambda_{YX}+\lambda_{HX})+4\pi(2M_Y+\lambda_G M_X)(\lambda_{YY}+\lambda_{HY}) \\ &\le 4\pi M_X(2\lambda_{YX}+\lambda_G\lambda_{XX}+\lambda_{G1}M_X) \\ &\quad +4\pi(2M_Y+\lambda_G M_X)(2\lambda_{YY}+\lambda_G\lambda_{XY})\end{aligned} \tag{VI.16}$$

And finally for R_3 we obtain the following estimate:

$$\begin{aligned}\|R_3\| &\le \lambda_G\|u\|+\|v\| \le 2\pi\lambda_G M_X+2\pi(M_Y+\|H\|) \\ &\le 4\pi\lambda_G M_X+4\pi M_Y\end{aligned} \tag{VI.17}$$

Considering the second equation in (VI.13) together with the corresponding initial conditions we can notice that the following equations are valid for η_1:

$$\begin{aligned}\dot{\eta}_1 &= \frac{dG(\xi_1,\varepsilon)}{dt}+\varepsilon^2\left(R_2-\frac{\partial G(\xi_1,\varepsilon)}{\partial \xi_1}R_1\right) \\ \eta_1(2\pi n) &= G(\xi_1(2\pi n),\varepsilon)+\varepsilon R_3,\ n=0,1,2,\ldots\end{aligned} \tag{VI.18}$$

This is a sequence of systems of differential equations. Each of them is valid in the time interval $2\pi n \le t < 2\pi(n+1)$ and has its own initial conditions. Integrating the system (VI.18) and taking the finiteness of each time interval into account the following relationship can be obtained:

$$\begin{aligned}\eta_1 &= G(\xi_1,\varepsilon)+\varepsilon R_4 \\ \|R_4\| &\le 2\pi\|R_3\| \le 8\pi^2(\lambda_G M_X+M_Y)\end{aligned} \tag{VI.19}$$

The last relation is valid for all times. Substituting (VI.19) into the first equation of the system (VI.13) and using the fact that the function Ξ also satisfies Lipschitz' conditions, we obtain the final form of the averaged equations:

$$\begin{aligned}\dot{\xi}_1 &= \varepsilon\Xi(\xi_1,G(\xi_1,\varepsilon),\varepsilon)+\varepsilon^2 R_5,\quad \xi_1(0)=x_0 \\ \eta_1 &= G(\xi_1,\varepsilon)+\varepsilon R_4\end{aligned} \tag{VI.20}$$

The following estimate is valid for the last residual R_5:

$$R_5 \le R_1 + R_4 \le 4\pi M_X \lambda_{XX} + 4\pi\left(2M_Y + \lambda_G M_X\right)\lambda_{XY} + 8\pi^2\left(\lambda_G M_X + M_Y\right) \tag{VI.21}$$

Notice that the system (VI.20) is equivalent to the original system (VI.1) at all times. Consider now the shortened system (VI.3) alongside the complete system (VI.20) and estimate the difference between their solutions:

$$\frac{d\left\|\xi_1 - \xi\right\|}{dt} \le \varepsilon\left(\lambda_{XX} + \lambda_{XY}\lambda_G\right)\left\|\xi_1 - \xi\right\| + \varepsilon^2 R_5 \tag{VI.22}$$

Applying the classical Gronwall's lemma we obtain the final estimate:

$$\left\|\xi_1 - \xi\right\| \le \varepsilon^2 R_5 t e^{\varepsilon\left(\lambda_{XX} + \lambda_{XY}\lambda_G\right)t} \tag{VI.23}$$

It guaranties the required asymptotic accuracy of the averaged system (VI.3). Explicit expressions for the constants C_1, C_2, C_3, C_4 can be easily obtained if one takes the estimations for the residuals into account. The proof is accomplished.

This theorem can not be applied directly to variable order discontinuous systems but it can be obviously generalized as follows.

Theorem 2
Consider following problem:

$$\begin{aligned}
&\dot{x} = \varepsilon X\left(x, yM_1(z,\alpha), \varphi, zM_1(z,\alpha), \varepsilon\right)\\
&x(0) = x_0\\
&\dot{y}M_1(z,\alpha) = \varepsilon Y\left(x, yM_1(z,\alpha), \varphi, zM_1(z,\alpha), \varepsilon\right)M_1(z,\alpha)\\
&y(t_n) = G\left(x(t_n)\right) + \varepsilon G_1\left(x(t_n), \varphi(t_n), \varepsilon\right)\\
&\dot{z}M_1(z,\alpha) = M_1(z,\alpha)\\
&\qquad + \varepsilon Z\left(x, yM_1(z,\alpha), \varphi, zM_1(z,\alpha), \varepsilon\right)M_1(z,\alpha)\\
&z(t_n) = 2\pi n\\
&\dot{\varphi} = 1 + \varepsilon\Phi\left(x, yM_1(z,\alpha), \varphi, zM_1(z,\alpha), \varepsilon\right)\\
&\varphi(t_n) = 2\pi n + \varepsilon F\left(x(t_n), \varphi(t_n), \varepsilon\right)
\end{aligned} \tag{VI.24}$$

Here x and y are, as before, vector functions of arbitrary finite dimensions, z and φ are scalar phases, the requirements on the functions X, Y and G are

the same as in the Theorem 1, the function F has in addition bounded partial derivatives with respect to its first two arguments.

The function $M_1(z,\alpha)$ is defined as follows (see Fig. AVI.1):

$$M_1(z,\alpha)=\begin{cases}1, & 0\le z<\alpha\\ 0, & \alpha\le z<2\pi\end{cases},\quad M(z+2\pi,\alpha)=M(z) \tag{VI.25}$$

The switch point α may depend on slow variables: $\alpha=\alpha(x,y,z-\varphi)$, but it should remain in the interval $\alpha\in(0,\ 2\pi)$. Otherwise function M_1 would not be defined correctly.

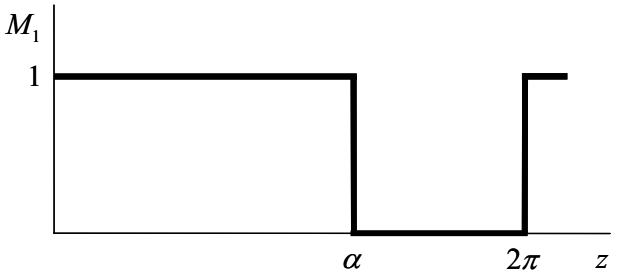

Fig. AVI.1. On the definition of the function M_1

Consider the averaged system alongside (VI.24):

$$\begin{aligned}&\dot{\xi}=\varepsilon\left\langle X\left(\xi,G(\xi,\varepsilon)M_1(\psi,\alpha),\psi,\psi M_1(\psi,\alpha),\varepsilon\right)\right\rangle_{\psi}\\ &\eta M_1(\psi,\alpha)=G(\xi,\varepsilon)M_1(\psi,\alpha)\\ &\varsigma M_1(\psi,\alpha)=\psi M_1(\psi,\alpha)\\ &\dot{\psi}=1\end{aligned} \tag{VI.26}$$

The solutions to (VI.24) and (VI.26) are asymptotically close to each other, i.e. the error one makes on using the averaged system instead of the original one is small for the asymptotically long time interval $t=O(1/\varepsilon)$:

$$\begin{aligned}&\|x-\xi\|=O(\varepsilon),\quad \|y-\eta\|M_1=O(\varepsilon)\\ &\|(z-\varphi)-(\varsigma-\psi)\|M_1=O(\varepsilon)\end{aligned} \tag{VI.27}$$

Proof

The proof of this theorem is almost trivial, because the system (VI.24) can be transformed to the form (VI.1).

A new fast rotating phase $\varphi_1 = \varphi - \varepsilon F$ and a new slow variable $\theta = z - \varphi_1$ can be introduced. The next step is to change to the phase

$$\varphi_2 = 2\pi n + \begin{cases} \dfrac{\pi(\varphi_1 - 2\pi n)}{\alpha - \theta}, & 2\pi n \le \varphi_1 \le 2\pi n + \alpha - \theta \\ \pi\left(1 + \dfrac{\varphi_1 - 2\pi n - (\alpha - \theta)}{2\pi - (\alpha - \theta)}\right), & 2\pi n + \alpha - \theta \le \varphi_1 \le 2\pi(n+1) \end{cases} \tag{VI.28}$$

The following discontinuous frequency corresponds to this new phase:

$$U = \begin{cases} \dfrac{\pi}{\alpha - \theta}, & 2\pi n \le \varphi_2 \le (2n+1)\pi \\ \dfrac{\pi}{2\pi - (\alpha - \theta)}, & (2n+1)\pi \le \varphi_2 \le 2\pi(n+1) \end{cases} \tag{VI.29}$$

Considering now φ_2 as the independent variable, the following system can be obtained:

$$\begin{aligned} x' &= \varepsilon \frac{X}{U} + O(\varepsilon^2) \\ \theta' M(\varphi_2) &= \left(\varepsilon \frac{Z - \Phi}{U} + O(\varepsilon^2)\right) M(\varphi_2) \\ y' M(\varphi_2) &= \left(\varepsilon \frac{Y}{U} + O(\varepsilon^2)\right) M(\varphi_2) \\ y(2\pi n) &= G(x(2\pi n), \varepsilon) + O(\varepsilon) \\ \theta(2\pi n) &= 0 \end{aligned} \tag{VI.30}$$

The required estimates (VI.27) can be obtained now applying Theorem 1 to the system (VI.30) and returning back to the original variables. The only important restriction is that during the whole time of the system's evolution the variable α is not allowed come in the ε-vicinity of zero or 2π. Otherwise the new phase φ_2 would become discontinuous. This circumstance is not connected with the proof procedure but it displays the important internal essence of the considered problem.

Appendix VII: Hierarchic Averaging in Systems with a Semi Slow Rotating Phase

(The following analysis is based on [86].)
Consider the following system

$$\begin{aligned} \dot{x} &= \varepsilon X\left(x,\alpha,t,\varepsilon\right) \\ \dot{\alpha} &= \sqrt{\varepsilon}A_1\left(x,\alpha,\varepsilon\right)+\varepsilon A_2\left(x,\alpha,t,\varepsilon\right) \end{aligned} \tag{VII.1}$$

Suppose

1. The functions X, A_1, A_2 are 2π -periodic with respect to t and α, bounded and satisfy together with their first partial derivatives Lipschitz-conditions with respect to the arguments x and α.
2. Variable α is the semi slow phase, i.e. the function A_1 is separated from zero:

$$A_1 \geq A_0 > 0,\ A_0 = const \tag{VII.2}$$

3. All other conditions are the same as in the first Bogoliubov's theorem for standard averaging.

Let us formulate the hierarchic averaging procedure. As usual, it is based on the almost identical transformation

$$\begin{aligned} x &= \xi_1 + \varepsilon u\left(\xi_1,\beta_1,t,\varepsilon\right) \\ \alpha &= \beta_1 + \sqrt{\varepsilon}w_1\left(\xi_1,\beta_1,t\right)+\varepsilon w_2\left(\xi_1,\beta_1,t,\varepsilon\right) \end{aligned} \tag{VII.3}$$

The corresponding equations governing the new variables ξ_1 and β_1 have the following form:

$$\begin{aligned} \dot{\xi}_1 &= \varepsilon\Xi\left(\xi_1,\beta_1,\varepsilon\right)+\varepsilon^{3/2}O\left(1\right) \\ \dot{\beta}_1 &= \sqrt{\varepsilon}B_1\left(\xi_1,\beta_1\right)+\varepsilon B_2\left(\xi_1,\beta_1,\varepsilon\right)+\varepsilon^{3/2}O\left(1\right) \end{aligned} \tag{VII.4}$$

These equations are obtained as the second order approximation for averaging with respect to t. The functions in (VII.4) are:

$$\begin{aligned}
&\Xi\left(\xi_1,\beta_1,\varepsilon\right)=\left\langle X\left(\xi_1,\beta_1,t,\varepsilon\right)\right\rangle_t\\
&B_1\left(\xi_1,\beta_1\right)=\left\langle A_1\left(\xi_1,\beta_1,t\right)\right\rangle_t\\
&w_1=\int_0^t\left\{A_1\left(\xi_1,\beta_1,\tau\right)-B_1\left(\xi_1,\beta_1\right)\right\}d\tau\\
&B_2\left(\xi_1,\beta_1,\varepsilon\right)=\left\langle A_2+\frac{\partial A_1}{\partial\beta_1}w_1\right\rangle_t
\end{aligned}\tag{VII.5}$$

It is necessary to notice, that B_1 is similarly to A_1 a function which is separated from zero and 2π -periodic with respect to β_1. It means that β_1 is the semi slow phase. The second step of the hierarchic averaging scheme is the averaging with respect to this phase, performed as usual by means of the following almost identical transformation:

$$\begin{aligned}
&\xi_1=\rho_2+\sqrt{\varepsilon}\,p\left(\rho_2,\gamma_2,\varepsilon\right)\\
&\beta_1=\gamma_2+\sqrt{\varepsilon}q\left(\rho_2,\gamma_2,\varepsilon\right)
\end{aligned}\tag{VII.6}$$

Here p and q are bounded 2π -periodic functions of γ_2 with zero average. The corresponding governing equations are:

$$\begin{aligned}
&\dot{\xi}_2=\varepsilon P\left(\rho_2,\varepsilon\right)+\varepsilon^{3/2}O\left(1\right)\\
&\dot{\gamma}_2=\sqrt{\varepsilon}B_1\left(\rho_2,\gamma_2,\varepsilon\right)+\varepsilon\Gamma_2\left(\rho_2,\varepsilon\right)+\varepsilon^{3/2}O\left(1\right)
\end{aligned}\tag{VII.7}$$

Here

$$\begin{aligned}
&P\left(\rho_2,\varepsilon\right)=\left\langle\frac{\Xi\left(\rho_2,\gamma_2,\varepsilon\right)}{B_1\left(\rho_2,\gamma_2\right)}\right\rangle_{\gamma_2}\\
&\Gamma_2\left(\rho_2,\varepsilon\right)=\left\langle\frac{\dfrac{\partial B_1}{\partial\rho_2}p+B_2\left(\rho_2,\gamma_2,\varepsilon\right)}{B_1{}^2\left(\rho_2,\gamma_2\right)}\right\rangle_{\gamma_2}\frac{1}{\left\langle\dfrac{1}{B_1\left(\rho_2,\gamma_2\right)}\right\rangle_{\gamma_2}}\\
&p\left(\rho_2,\gamma_2,\varepsilon\right)=\int_{\gamma_0}^{\gamma_2}\frac{\Xi\left(\rho_2,\gamma,\varepsilon\right)-P\left(\rho_2,\varepsilon\right)}{B_1\left(\rho_2,\gamma\right)}d\gamma
\end{aligned}\tag{VII.8}$$

The system (VII.7) is absolutely equivalent to the original system (VII.1). Now we are going to consider the shortened system

$$\begin{aligned}\dot{\xi} &= \varepsilon P\left(\rho,\varepsilon\right)\\ \dot{\gamma} &= \sqrt{\varepsilon}B_1\left(\rho,\gamma,\varepsilon\right)+\varepsilon\Gamma_2\left(\rho,\varepsilon\right)\end{aligned} \tag{VII.9}$$

Notice that we have averaged only the equations for ξ. The equation for γ still contains γ, i.e. the system (VII.9) is not completely autonomous.

The following theorem estimates the difference between the solutions to the full and shortened systems.

Theorem

Under the formulated conditions the following estimates fir the relationship between the solutions of the systems (VII.7) and (VII.9) are valid for the time interval $O\left(1/\varepsilon\right)$:

$$\begin{aligned}\left\|x-\rho\right\| &\le C_{\rho 1}\sqrt{\varepsilon}\\ \left\|\alpha-\gamma\right\| &\le C_{\gamma 1}\end{aligned} \tag{VII.10}$$

Notice that the mistake in the phase is not small but limited. The mistake in the slow variables remains small at the very long time interval.

Proof

The difference $\left\|\rho-\rho_2\right\|$ can be estimated by means of the Gronwall's lemma quite easily:

$$\begin{aligned}\left\|\rho-\rho_2\right\| &\le \varepsilon\int_0^t\left\|P\left(\rho,\varepsilon\right)-P\left(\rho_2,\varepsilon\right)\right\|dt+\varepsilon^{3/2}O\left(1\right)t\\ &\le \varepsilon\lambda_P\int_0^t\left\|\rho-\rho_2\right\|\ dt+\varepsilon^{3/2}O\left(1\right)t\\ \Rightarrow \left\|\rho-\rho_2\right\| &\le \varepsilon^{3/2}O\left(1\right)te^{\varepsilon\lambda_P t}\end{aligned} \tag{VII.11}$$

This estimation guarantees the accuracy of $O\left(\sqrt{\varepsilon}\right)$ for the variable ρ on the time interval $t\sim O\left(1/\varepsilon\right)$.

In order to estimate the difference $\left\|\gamma-\gamma_2\right\|$ one can divide the corresponding equations (VII.7) and (VII.9):

$$\frac{d\gamma}{\Gamma(\rho,\gamma,\varepsilon)}=\frac{d\gamma_2}{\Gamma(\rho_2,\gamma_2,\varepsilon)+\varepsilon O(1)} \qquad \text{(VII.12)}$$
$$\Gamma(\rho,\gamma,\varepsilon)\equiv\Gamma_1(\rho,\gamma,\varepsilon)+\sqrt{\varepsilon}\Gamma_2(\rho,\varepsilon)$$

According to (VII.11) on the considered time interval $\rho_2=\rho+\sqrt{\varepsilon}O(1)$, function Γ is bounded, differentiable and separated from zero. Thus the relationship (VII.12) can be rewritten as follows:

$$\frac{d\gamma}{\Gamma(\rho_2,\gamma,\varepsilon)}+\sqrt{\varepsilon}O(1)d\gamma=\frac{d\gamma_2}{\Gamma(\rho_2,\gamma_2,\varepsilon)}+\varepsilon O(1)d\gamma_2 \qquad \text{(VII.13)}$$

The variables γ and γ_2 satisfy the same initial condition: $\gamma(0)=\gamma_2(0)=\gamma_0$. Thus the last equation can be integrated:

$$\int_{\gamma_0}^{\gamma_2}\frac{dg}{\Gamma(\rho_2,g,\varepsilon)}-\int_{\gamma_0}^{\gamma}\frac{dg}{\Gamma(\rho_2,g,\varepsilon)}=\int_{\gamma}^{\gamma_2}\frac{dg}{\Gamma(\rho_2,g,\varepsilon)} \qquad \text{(VII.14)}$$
$$=\sqrt{\varepsilon}O(1)(\gamma-\gamma_0)-\varepsilon O(1)(\gamma_2-\gamma_0)$$

Due to the condition, that the function Γ is separated from zero, the last relationship leads directly to the following estimate:

$$\|\gamma-\gamma_2\|\le\sqrt{\varepsilon}O(1)\|\gamma-\gamma_0\| \qquad \text{(VII.15)}$$

It is clear from the last equation in (VII.9) that $\|\gamma-\gamma_0\|\le\sqrt{\varepsilon}tO(1)$. Consequently

$$\|\gamma-\gamma_2\|\le\varepsilon tO(1) \qquad \text{(VII.16)}$$

Taking now the transformations (VII.3) and (VII.6) into account we obtain the required estimates for the time interval $O(1/\varepsilon)$ and accomplish the proof of the theorem.

Remark 1

The quantitative estimates of the mistakes can be obtained totally similar to those in Appendix VI.

Remark 2

The described hierarchic averaging can be combined with all the averaging procedures for discontinuous systems discussed in Appendix III – Appendix VI.

Appendix VIII: Averaging in Systems with Strong High Frequency Excitation

(The following analysis is based on [34].)

Theorem
Consider the system

$$\ddot{x} = F\left(x,\dot{x},t,\tau\right) + \omega\Phi\left(x,\dot{x},t,\tau\right),\ \ \tau \equiv \omega t$$
$$x\big|_{t=0} = x_0,\ \dot{x}\big|_{t=0} = v_0;\quad \varepsilon = 1/\omega << 1 \tag{VIII.1}$$

Suppose:

1. The functions F and Φ together with their first partial derivatives are 2π - periodic with respect to τ, bounded and satisfy Lipschitz-conditions with respect to the arguments x and $\dot{x}$.
2. The functions F and Φ are measurable functions with respect to t and τ.
3. The function Φ has in addition bounded and Lipschitz'-continuous second partial derivatives with respect to the arguments x and $\dot{x}$.
4. The symbol $\langle\ \rangle_\tau$ means averaging with respect to τ.
5. $\omega >> 1$ is the large parameter.
6. The general 2π - periodic with respect to τ bounded solution to the system of n first order differential equations is known:

$$\frac{\partial u}{\partial \tau} = \Phi\left(X,u,t,\tau\right),\ \left\langle u\left(X,t,\tau\right)\right\rangle_\tau = \dot{X} \Rightarrow u = U\left(X,\dot{X},t,\tau\right) \tag{VIII.2}$$

This solution satisfies the Lipschitz' conditions together with its first partial derivatives with respect to X and $\dot{X}$.

7. The fundamental matrix of solutions for the following linear homogeneous system is known, bounded and its determinant does not depend on ε and is not equal to zero

$$\frac{\partial W_*}{\partial \tau} = -\left(\frac{\partial \Phi}{\partial \dot{x}}\right)^T W_*;\ \ \det\left\langle W_*^T\right\rangle \neq 0 \tag{VIII.3}$$

8. The following determinant is not equal to zero:

$$\det\left\{\left\langle W_*^T \frac{\partial U}{\partial \dot{X}}\right\rangle_\tau\right\} = 0 \tag{VIII.4}$$

Consider together with (VIII.1) a system of ordinary differential equation, which does not contain the fast time τ :

$$M\left(X,\dot{X},t\right)\ddot{X}=V\left(X,\dot{X},t\right);$$
$$X\big|_{t=0}=x_0,\dot{X}\big|_{t=0}=v_0-\partial\Psi/\partial\tau\big|_{t=0,\tau=0} \tag{VIII.5}$$

$$M\left(X,\dot{X},t\right)=\left\langle W_*^T\frac{\partial U}{\partial\dot{X}}\right\rangle_\tau;\qquad \Psi=\int_0^\tau\left(U-\dot{X}\right)d\tau$$
$$V\left(X,\dot{X},t\right)=\left\langle W_*^T F\left(X,U,t,\tau\right)\right\rangle_\tau \tag{VIII.6}$$
$$+\left\langle W_*^T\left\{\frac{\partial\Phi}{\partial X}\Psi-\frac{\partial U}{\partial X}\dot{X}-\frac{\partial U}{\partial t}\right\}\right\rangle_\tau$$

Under these assumptions the difference between the solutions of the original system (VIII.1) and the averaged system (VIII.5), (VIII.6) remains small for the asymptotic long time interval with respect to the fast time $\tau=O\left(\omega\right)$.

Proof

We perform the following transformation:

$$x=\xi\left(t\right)+\varepsilon v\left(\xi,\eta,t,\tau\right),\ \dot{\xi}=\Xi\left(\xi,\eta,t\right)$$
$$\dot{x}=\eta\left(t\right)+\frac{\partial\psi\left(\xi,\eta,t,\tau\right)}{\partial\tau}+\varepsilon w\left(\xi,\eta,t,\tau\right),\ \dot{\eta}=\mathrm{N}\left(\xi,\eta,t\right) \tag{VIII.7}$$

The functions ψ,v,w,Ξ,N have to be chosen in order to obtain the "slow" governing equations for the variables ξ and η. The result is:

$$\Xi+\varepsilon\frac{\partial v}{\partial\xi}\Xi+\varepsilon\frac{\partial v}{\partial\eta}\mathrm{N}+\varepsilon\frac{\partial v}{\partial t}+\frac{\partial v}{\partial\tau}=\eta+\frac{\partial\psi}{\partial\tau}+\varepsilon w \tag{VIII.8}$$

$$\mathrm{N}+\frac{\partial^2\psi}{\partial\xi\partial\tau}\Xi+\frac{\partial^2\psi}{\partial\eta\partial\tau}\mathrm{N}+\frac{\partial^2\psi}{\partial t\partial\tau}+\frac{1}{\varepsilon}\frac{\partial^2\psi}{\partial\tau^2}+\varepsilon\frac{\partial w}{\partial\xi}\Xi+\varepsilon\frac{\partial w}{\partial\eta}\mathrm{N}$$
$$+\varepsilon\frac{\partial w}{\partial t}+\frac{\partial w}{\partial\tau}=F\left(\xi+\varepsilon v,\eta+\frac{\partial\psi}{\partial\tau}+\varepsilon w,t,\tau\right) \tag{VIII.9}$$
$$+\frac{1}{\varepsilon}\Phi\left(\xi+\varepsilon v,\eta+\frac{\partial\psi}{\partial\tau}+\varepsilon w,t,\tau\right)$$

Balancing the terms of the same magnitude order one obtains:

$$\frac{\partial^2\psi}{\partial\tau^2}=\Phi\left(\xi,\eta+\frac{\partial\psi}{\partial\tau},t,\tau\right) \tag{VIII.10}$$

$$\begin{aligned}&\Xi+\frac{\partial u}{\partial\tau}=\eta+\frac{\partial\psi}{\partial\tau}\\&\mathrm{N}+\frac{\partial^2\psi}{\partial\xi\partial\tau}\Xi+\frac{\partial^2\psi}{\partial\eta\partial\tau}\mathrm{N}+\frac{\partial^2\psi}{\partial t\partial\tau}+\frac{\partial w}{\partial\tau}\\&\quad=F\left(\xi,\eta+\frac{\partial\psi}{\partial\tau},t,\tau\right)+\left.\frac{\partial\Phi}{\partial x}\right|_{\xi,\eta+\frac{\partial\psi}{\partial\tau}}v+\left.\frac{\partial\Phi}{\partial\dot{x}}\right|_{\xi,\eta+\frac{\partial\psi}{\partial\tau}}w+\varepsilon R_1\end{aligned} \tag{VIII.11}$$

Here $R_1=O(1)$ is a residual which can be estimated due to the assumptions 1 and 3.

Consider the equation (VIII.10). It can be transformed to the following form:

$$\frac{\partial u}{\partial\tau}=\Phi\left(\xi,u,t,\tau\right),\ u=\eta+\frac{\partial\psi}{\partial\tau} \tag{VIII.12}$$

The function ψ must be periodic with respect to τ. Thus the average $\left\langle\partial\psi/\partial\tau\right\rangle_\tau=0$. According to (VIII.2) the solution to the system (VIII.12) is known:

$$\begin{aligned}&u=U\left(\xi,\eta,t,\tau\right)\\&\psi=\Psi=\int_0^\tau\left(U-\dot{X}\right)d\tau\end{aligned} \tag{VIII.13}$$

Consider the equations (VIII.11). The first of them is quite simple to solve:

$$\Xi=\eta;\ v=\Psi+V\left(t\right) \tag{VIII.14}$$

Here $V\left(t\right)$ is an arbitrary function of the slow time t. Then the second of the equations (VIII.11) can be rewritten as follows:

$$\frac{\partial w}{\partial \tau}=\frac{\partial \Phi}{\partial \dot{x}}w+F-\mathrm{N}+\frac{\partial \Phi}{\partial x}\left(\Psi+V\right)-\left(\frac{\partial^2\Psi}{\partial\xi\partial\tau}\Xi+\frac{\partial^2\Psi}{\partial\eta\partial\tau}\mathrm{N}+\frac{\partial^2\Psi}{\partial t\partial\tau}\right)+\varepsilon R_1 \tag{VIII.15}$$

Notice that the expression in brackets is the derivative with respect to τ from the full partial derivative of $\Psi\left(t,\tau\right)$ with respect to t (taking in account that Ψ depends on t both directly and indirectly through ξ and η). Thus one can rewrite (VIII.15):

$$\frac{\partial w}{\partial \tau}=\frac{\partial \Phi}{\partial \dot{x}}w+F-\mathrm{N}+\frac{\partial \Phi}{\partial x}\left(\Psi+V\right)-\frac{\partial^2\Psi}{\partial t\partial\tau}+\varepsilon R_1 \tag{VIII.16}$$

The necessary condition for the existence of the periodic solution to (VIII.16) is (*cf.* [98]):

$$\left\langle W_*^T\left\{F-\mathrm{N}+\frac{\partial \Phi}{\partial x}\left(\Psi+V\right)-\frac{\partial^2\Psi}{\partial t\partial\tau}+\varepsilon R_1\right\}\right\rangle_\tau=0 \tag{VIII.17}$$

The matrix W_* is defined in (VIII.3). It was shown in Chapter 9 that

$$\left\langle W_*^T\frac{\partial \Phi}{\partial x}\right\rangle_\tau=0 \tag{VIII.18}$$

Hence the equation (VIII.17) together with (VIII.14) can be rewritten as follows:

$$\left\langle W_*^T\right\rangle_\tau\ddot{\xi}=\left\langle W_*^T\left\{F+\frac{\partial \Phi}{\partial x}\Psi-\frac{\partial^2\Psi}{\partial t\partial\tau}\right\}\right\rangle_\tau+\varepsilon\left\langle W_*^T R_1\right\rangle_\tau \tag{VIII.19}$$

Lastly, function Ψ depends on t both directly and indirectly through functions $X\left(t\right)$ and $\dot{X}(t)$. Under $\partial/\partial t$ we understand here the "full" partial derivative with respect to t. Taking this into account and using the "partial" partial derivatives we obtain the final explicit form of the equation (VIII.19):

$$\frac{\partial \Psi}{\partial \tau} = U + \eta, \eta = \dot{\xi} \Rightarrow \frac{\partial^2 \Psi}{\partial t \partial \tau} = \frac{\partial U}{\partial t} + \frac{\partial U}{\partial \xi}\dot{\xi} + \frac{\partial U}{\partial \dot{\xi}}\ddot{\xi} + \ddot{\xi} \Rightarrow$$

$$\left\langle W_*^T \frac{\partial U}{\partial \dot{\xi}} \right\rangle_\tau \ddot{\xi} = \left\langle W_*^T \left\{ F + \frac{\partial \Phi}{\partial x}\Psi - \frac{\partial U}{\partial t} - \frac{\partial U}{\partial \xi}\dot{\xi} \right\} \right\rangle_\tau \qquad \text{(VIII.20)}$$

$$+ \varepsilon \left\langle W_*^T R_1 \right\rangle_\tau$$

It is easy to show that all the functions here are limited and Lipschitz-continuous. Thus we can apply the Gronwall's lemma in order to compare the solutions to the full system (VIII.20) and to the shortened equations (VIII.5), (VIII.6) and obtain the required estimate.

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